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# THE GEOMETRY OF SYMMETRIC SPACES AND CLIFFORD ALGEBRAS

by

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Adrian Banner

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# Abstract

This thesis presents a unified construction of rank one symmetric spaces of non-compact type. These spaces correspond to hyperbolic geometries of real, complex, quaternionic and octonionic types. We extend the approach of Cowling, Dooley, Korányi and Ricci using algebras of Heisenberg type, Clifford algebras and Spin groups by emphasising the explicit geometry of these spaces.

Our concrete approach allows us to give a new proof of a result due to Pansu on graded automorphisms of certain Lie algebras. We also prove a conjecture of Korányi concerning metrics on the boundary of the symmetric spaces and demonstrate that the classical Cayley transform extends to a 1-quasiconformal map of the boundary.

# Contents

# Introduction

The symmetric spaces of rank one of noncompact type are often grouped into four families, corresponding to hyperbolic geometries of real, complex, quaternionic and octonionic types. Certain results hold for all four families, yet due to the different properties of the underlying division algebra (in particular the nonassociativity of the octonions) the proofs of these results often require an examination of several cases. Not only does this require more effort, it can also obscure the underlying reasons for the truth of the results. An important example is Mostow's rigidity theorem, which holds for all four families but is proved (see [M]) with considerably greater difficulty in the octonionic case.

We therefore seek a more unified way of approaching the symmetric spaces in question. In [CDKR] and [CDKR2], the spaces are formulated in a new way using algebras of Heisenberg type. In this thesis we continue this formulation and present several results which utilise it.

We begin with a review of elementary hyperbolic geometry. In Chapter 1 we construct several models of real and complex hyperbolic space in order to provide motivation for the method of construction of the general symmetric spaces. Much of this material is derived from the author's Honours thesis [B]. We use a differential geometrical approach in order to highlight the similarities between the special cases of real and complex hyperbolic spaces and the general spaces considered in Chapter 3.

In Chapter 2 we develop most of the algebra we shall need. In particular, we review some elementary properties of quaternions and octonions. The connection with Clifford algebras is highlighted particularly in the case of the triality automorphism of $Spin(8)$ which itself is intrinsically linked with the octonions. The mathematics in this chapter is well-known, however the presentation is somewhat nontraditional and is based on material from [Ps], many of the details of which have been clarified and reworked for this thesis. In particular the triality automorphism is constructed from its action on the roots of $\mathfrak{so}(8)$ and then used to define the octonions. This represents a reversal of the usual order of presentation of these topics, in which the octonions are used to define the triality automorphism.

The definition and properties of algebras of Heisenberg type ($H$-type algebras) are presented in Section 1 of Chapter 3. These algebras are generalisations of the Lie algebras of the classical Heisenberg groups. After extending an $H$-type algebra by adjoining a one-dimensional subspace, we isolate two (overlapping) regions of the resulting vector space and equip them with Riemannian metrics. The first region is the unit ball whereas the second is a Siegel-type domain. There is a map known as the Cayley transform which isometrically identifies the two resulting Riemannian manifolds. In [CDKR] and [CDKR2] it is proved that there is a bijection between the set of symmetric spaces of rank one of noncompact type and $H$-type algebras satisfying a condition known as the $J^2$ condition. In fact both the unit ball model and the Siegel-type model are symmetric spaces if and only if the underlying $H$-type algebra satisfies the $J^2$ condition. In this case the two models are effectively generalisations of the disc (Klein) model and the upper-half-space model of real hyperbolic space. In Section 3.3 we show explicitly how all four families of the symmetric spaces may be modelled by the unit ball or Siegel-type domains. We also exhibit a connection between various subgroups of the group of isometries of the symmetric spaces and Spin groups. Using our unified approach, we then give a new proof of Pansu's result ([Pu], Proposition 10.1) describing the automorphisms of the symmetric spaces belonging to the quaternionic and octonionic families.

In the final chapter we extend the construction of the symmetric spaces by identifying the geodesics and calculating the distance formulae. Although these are well-known, particularly in the real and complex cases, once again we are able to highlight the uniformity of the approach by describing distances in all of the rank one symmetric spaces of noncompact type using either of two equivalent formulae.

Having considered the geometry of the symmetric spaces, we examine the boundary "at infinity". The properties of certain classes of functions on the symmetric spaces are governed by their behaviour of their extensions to the boundary. In particular, isometries of the symmetric spaces extend to 1-quasiconformal maps of the boundary. This fact is pivotal in the proof of Mostow's rigidity theorem. In Section 2 of Chapter 4 we define a function of two variables on the unit sphere of an extended $H$-type algebra, the sphere being considered as the boundary of the unit ball model. This function was conjectured by Korányi to be a distance function when the $J^2$ condition holds in the underlying $H$-type algebra. We prove that this is indeed the case and furthermore that the condition is also necessary. We also show that the Cayley transform identifying the unit ball model and the Siegel-type model extends continuously to the boundary, where it is 1-quasiconformal with respect to Korányi's metric on the sphere and a standard metric on the boundary of the Siegel-type domain. This implies that all properties concerning quasiconformality of the boundary may be examined in either model by using the Cayley transform to pass from one model to the other.

# Chapter 1

# Hyperbolic Geometry

The simplest of the four families of symmetric spaces of rank one of noncompact type consists of the spaces $O_0(1,n)/O(n)$ for $n \geq 1$. In this chapter we present a geometrical treatment of these spaces, treating them as models of real hyperbolic geometry. We also investigate the next most simple family, consisting of the spaces $U(1,n)/U(n)$ for $n \geq 1$, in a similar way. Many of the ideas presented in this chapter will be generalised in later chapters. Most of the material in this chapter has been adapted from [B].

## 1.1 Real Hyperbolic $n$-space

In this section, we examine a model of hyperbolic geometry which we regard as being the definition of real hyperbolic $n$-space. The underlying space is a hyperboloid of revolution, which is equipped with an appropriate Riemannian metric. We investigate the properties of the group of isometries, the geodesics and the associated distance function on the resultant manifold.

**Definition** The *Lorentzian form* $\langle \cdot , \cdot \rangle$ on $\mathbf{R}^{n+1}$ is given by

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n \tag{1.1}$$

for all $x = (x_0, \ldots, x_n)$, $y = (y_0, \ldots, y_n) \in \mathbf{R}^{n+1}$. Define *real hyperbolic n-space $H^n$* by

$$H^n = \left\{ x \in \mathbf{R}^{n+1} : \langle x, x \rangle = -1 \text{ and } x_0 > 0 \right\}.$$

Geometrically, $H^n$ is one of the two sheets of the hyperboloid

$$x_0^2 - x_1^2 - \cdots - x_n^2 = 1$$

in $\mathbf{R}^{n+1}$. In order to make $H^n$ into a Riemannian manifold, we shall use the following lemma.

**Lemma 1.1** *Let* $x, y \in \mathbf{R}^{n+1} \setminus \{0\}$ *such that* $\langle x, x \rangle < 0$ *and* $\langle x, y \rangle = 0$. *Then* $\langle y, y \rangle > 0$.

**Proof** Since $\langle x, x \rangle < 0$, we have

$$x_0^2 > x_1^2 + \cdots + x_n^2,$$

hence $x_0 \neq 0$. The condition $\langle x, y \rangle = 0$ may be expressed as

$$x_0 y_0 = \sum_{i=1}^{n} x_i y_i. \tag{1.2}$$

By the Cauchy–Schwartz inequality,

$$\begin{aligned} x_0^2 \langle y, y \rangle &= x_0^2 \left(-y_0^2 + y_1^2 + \cdots + y_n^2\right) = -\left(\sum_{i=1}^{n} x_i y_i\right)^2 + x_0^2 \sum_{i=1}^{n} y_i^2 \\ &\geq -\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) + x_0^2 \sum_{i=1}^{n} y_i^2 = \left(\sum_{i=1}^{n} y_i^2\right)\left(x_0^2 - x_1^2 - \cdots - x_n^2\right) \\ &\geq 0, \end{aligned}$$

hence $\langle y, y \rangle \geq 0$. Furthermore, if $\langle y, y \rangle = 0$, the above inequalities imply that $\sum_{i=1}^{n} y_i^2 = 0$, whence $y_1 = \cdots = y_n = 0$. By (1.2), $y_0 = 0$, contradicting the assumption that $y \neq 0$, hence $\langle y, y \rangle > 0$ as claimed. $\square$

The tangent space at the point $x \in H^n$ may be identified with the set of all vectors $y \in \mathbf{R}^{n+1}$ such that $y$ is tangential to $H^n$ when $y$ is considered to be based at $x$. In particular, we must have $y \cdot \nabla f(x) = 0$, where $f : \mathbf{R}^{n+1} \to \mathbf{R}$ is defined by $f(x) = \langle x, x \rangle$ for all $x \in \mathbf{R}$. This condition is trivially equivalent to $\langle x, y \rangle = 0$, thus we identify the tangent space $T_x H^n$ at $x$ with

$$\left\{y \in \mathbf{R}^{n+1} : \langle x, y \rangle = 0\right\},$$

which is an $n$-dimensional subspace of $\mathbf{R}^{n+1}$. Furthermore, for each $x \in H^n$, we define an inner product on $T_xH^n$ by

$$\langle y, y \rangle_x = \langle y, y \rangle$$

for all $y \in T_xH^n$. Lemma 1.1 implies that the associated Riemannian metric is positive definite.

**Definition** The *Lorentz group* of $\mathbf{R}^{n+1}$ is given by

$$\begin{aligned} O(1,n) &= \{P \in GL(n+1, \mathbf{R}) : \langle Px, Py \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbf{R}^{n+1}\} \\ &= \{P \in GL(n+1, \mathbf{R}) : P^t S P = S\} \end{aligned}$$

where $P^t$ denotes the transpose of $P$,

$$S = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$$

and $I_n$ denotes the $n \times n$ identity matrix. The *positive Lorentz group* of $\mathbf{R}^{n+1}$ is given by

$$PO(1,n) = \left\{ P \in O(1,n) : \frac{(Px)_0}{x_0} > 0 \text{ whenever } \langle x, x \rangle < 0 \right\}.$$

In fact $PO(1,n)|_{H^n} = I(H^n)$, the group of isometries of $H^n$. That is, if $f : H^n \to H^n$ is any isometry then $f$ is the restriction to $H^n$ of some $P \in PO(1,n)$, and conversely, if $P \in PO(1,n)$, then $P|_{H^n} \in I(H^n)$. Furthermore, the action of $I(H^n)$ on $H^n$ is transitive. To see this, let

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} : R \in O(n) \right\}$$

and

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\}.$$

Clearly $K$ and $A$ are subgroups of $PO(1,n)$ and $K$ is the stabiliser of $(1, 0, \ldots, 0)$. Given $x, y \in H^n$, we may find $k_1, k_2 \in K$ such that

$$k_1(x) = (x_0, 0, \ldots, 0, x_n') \qquad \text{and} \qquad k_2(y) = (y_0, 0, \ldots, 0, y_n'),$$

where $x_0^2 - (x_n')^2 = y_0^2 - (y_n')^2 = 1$. There exists $a \in A$ such that

$$a(x_0, 0, \dots, 0, x_n') = (y_0, 0, \dots, 0, y_n').$$

(In particular, $a = a_t$ where $t$ satisfies $x_0 \cosh t + x_n' \sinh t = y_0$.) Setting $g = k_2^{-1} a k_1$, we see that $g(x) = y$ as required. We thus have the *Cartan decomposition* of $I(H^n)$,

$$I(H^n) = KAK,$$

where it is understood that $K$ and $A$ act on $H^n$ by restriction of domain. (Proofs of all unverified claims in the above paragraph may be found in [R] or [C].)

We now find the geodesics and associated distance function on $H^n$.

**Lemma 1.2** *If $x = (x_0, 0, \dots, 0, x_n) \in H^n$, $x_0 \neq 1$, and $y = (1, 0, \dots, 0)$, then the geodesic through $x$ and $y$ is the intersection of the plane $\{(a, 0, \dots, 0, b) : a, b \in \mathbf{R}\}$ of $\mathbf{R}^{n+1}$ with $H^n$. The geodesic may be parametrised as a unit speed curve by*

$$\gamma(t) = (\cosh t, 0, \dots, 0, \sinh t)$$

*where $t \in \mathbf{R}$. Furthermore, the length of the geodesic arc $\alpha$ joining $x$ and $y$ is given by*

$$|\alpha| = \cosh^{-1} x_0.$$

**Proof** The proof that the geodesic is the intersection with $H^n$ of the given plane may be found in Section 1.2.1 (although the fact may seem evident from symmetry considerations alone). The given curve $\gamma$ is a unit speed curve, since

$$\langle \dot\gamma(t), \dot\gamma(t) \rangle = -\sinh^2 t + \cosh^2 t = 1.$$

Furthermore $\gamma(0) = y$ and $\gamma(t_0) = x$, where $t_0 = \sinh^{-1} x_n$. It follows that if $\alpha$ is the restriction of $\gamma$ to $[0, \sinh^{-1} x_n]$ (or $[\sinh^{-1} x_n, 0]$ as appropriate), then

$$|\alpha| = \left|\sinh^{-1} x_n\right| = \cosh^{-1} x_0$$

since $x_0^2 - x_n^2 = 1$. □

**Theorem 1.3** *Let $x, y \in H^n$ with $x \neq y$. The geodesic through $x$ and $y$ is the intersection with $H^n$ of the plane through $x$, $y$ and the origin of $\mathbf{R}^{n+1}$. The length of the geodesic arc $\alpha$ joining $x$ to $y$ (that is, the associated Riemannian distance function) is given by*

$$d(x, y) = |\alpha| = \cosh^{-1}\left(-\langle x, y \rangle\right).$$

**Proof** Let $x, y \in H^n$. Using the Cartan decomposition of $I(H^n)$, there exists $g \in AK$ such that $g(y) = (1, 0, \dots, 0)$. There exists a map $k \in K$ such that $k(g(x)) = (x'_0, 0, \dots, 0, x'_n)$, where $(x'_0)^2 - (x'_n)^2 = 1$. Since $K$ stabilises the point $(1, 0, \dots, 0)$, we see that $h(y) = (1, 0, \dots, 0)$ and $h(x) = (x'_0, 0, \dots, 0, x'_n)$, where $h = k \circ g \in I(H^n)$. By Lemma 1.2, the geodesic through $h(y)$ and $h(x)$ is the intersection with $H^n$ of the plane through $h(y)$, $h(x)$ and the origin of $\mathbf{R}^{n+1}$. Now $h^{-1} \in I(H^n)$ preserves planes through the origin and also preserves geodesics, thus the geodesic through $x$ and $y$ is the intersection with $H^n$ of the plane through $x$, $y$ and the origin, as claimed. Furthermore, Lemma 1.2 implies that

$$d\left(h(x), h(y)\right) = \cosh^{-1}\left(h(x)\right)_0 = \cosh^{-1}\left(-\langle h(x), h(y)\rangle\right);$$

since $h$ is an isometry, we have

$$d(x, y) = d(h(x), h(y)) = \cosh^{-1}\left(-\langle h(x), h(y)\rangle\right) = \cosh^{-1}\left(-\langle x, y\rangle\right)$$

as required. □

## 1.2 Models of Real Hyperbolic Space

The space $H^n$ endowed with the hyperbolic metric is technically only a model of real hyperbolic $n$-space, which is more correctly defined as the simply connected, connected $n$-dimensional real Riemannian manifold of constant curvature $-1$, unique up to isomorphism. In this section we examine three other models of real hyperbolic $n$-space.

### 1.2.1 The Projective Disc Model

The projective disc model, otherwise known as the Klein model, takes $B^n$, the unit ball of $\mathbf{R}^n$, to be the underlying space for $n$-dimensional real hyperbolic geometry. The geodesics are Euclidean line segments, although the Riemannian metric differs markedly from the Euclidean metric except at the origin.

**Definition** The *gnomonic projection* $\mu : H^n \to B^n$ is defined by

$$\mu(x_0, x_1, \dots, x_n) = \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

for all $x = (x_0, \dots, x_n) \in H^n$. Note that $\mu$ is well-defined, since $x_0^2 - x_1^2 - \cdots - x_n^2 = 1$ implies that $x_1^2 + \cdots + x_n^2 < x_0^2$. It is also a bijection (since $x_0 > 0$), with inverse given by

$$\mu^{-1}(y_1, \dots, y_n) = \frac{1}{\sqrt{1 - y_1^2 - \cdots - y_n^2}}(1, y_1, \dots, y_n) = \frac{1}{\sqrt{1 - |y|^2}}(1, y_1, \dots, y_n)$$

for all $y = (y_1, \dots, y_n) \in B^n$. Geometrically, $\mu(x)$ is the intersection of the line joining the origin of $\mathbf{R}^{n+1}$ to $x$ with the hyperplane $\{(1, y_1, \dots, y_n) : y_1, \dots, y_n \in \mathbf{R}\}$. The unit ball $B^n$ may be thought of as being embedded in $\mathbf{R}^{n+1}$ by identifying $y \in B^n$ with $(1, y) \in \mathbf{R}^{n+1}$.

We use $\mu^{-1}$ to transfer the Riemannian metric of $H^n$ to $B^n$ by requiring that $\mu$ be an isometry. A routine calculation shows that the Jacobian of $\mu^{-1}$ at the point $x = (x_1, \dots, x_n) \in B^n$ is given by

$$J = \frac{1}{\lambda^3} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \lambda^2 + x_1^2 & x_1 x_2 & \cdots & x_1 x_n \\ x_2 x_1 & \lambda^2 + x_2^2 & \cdots & x_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n x_1 & x_n x_2 & \cdots & \lambda^2 + x_n^2 \end{pmatrix}$$

where $\lambda = \sqrt{1 - x_1^2 - \cdots - x_n^2} = \sqrt{1 - |x|^2}$. If $y \in T_x B^n$, one calculates that

$$Jy = \frac{x \cdot y}{\lambda^3}(1, x) + \frac{1}{\lambda}(0, y)$$

and that

$$\langle Jy, Jy \rangle = \frac{(x \cdot y)^2}{\lambda^4} + \frac{|y|^2}{\lambda^2}.$$

If we transfer the metric using the obvious formula

$$\langle y, y \rangle_x = \langle \dot{\nu}(y), \dot{\nu}(y) \rangle_{\nu(x)},$$

where $\nu = \mu^{-1}$, then we have

$$\langle y, y \rangle_x = \frac{(1 - |x|^2)|y|^2 + (x \cdot y)^2}{(1 - |x|^2)^2}.$$

Alternatively, if we express $y = y_{\mathrm{rad}} + y_{\mathrm{tan}}$, where $y_{\mathrm{rad}} \parallel x$ and $y_{\mathrm{tan}} \perp x$, then

$$\langle y, y \rangle_x = \frac{|y_{\mathrm{rad}}|^2}{(1 - |x|^2)^2} + \frac{|y_{\mathrm{tan}}|^2}{1 - |x|^2}.$$

We call this metric the *Klein metric* on $B^n$. In order to find the geodesics of $B^n$ endowed with this metric, we first complete the proof of Lemma 1.2 using the following lemma.

**Lemma 1.4** *If $x = (0, \dots, 0, x_n) \in B^n$, with $x_n \neq 0$, then the (Kleinian) geodesic joining $y = (0, 0, \dots, 0)$ and $x$ is contained in the line segment*

$$\{(0, \dots, 0, a) : a \in [-1, 1]\}.$$

**Proof** Let $\beta : [a, b] \to B^n$ be any differentiable curve such that $\beta(a) = y$ and $\beta(b) = x$. Write $\beta(t) = \beta_1(t) + \beta_2(t)$, where $\beta_1(t) = (0, \dots, 0, |\beta(t)|)$ for all $t \in [a, b]$. A simple calculation shows that $|\beta(t)| = |\beta_1(t)|$ and $\beta(t) \cdot \dot{\beta}(t) = \beta_1(t) \cdot \dot{\beta}_1(t)$ for all $t \in [a, b]$; furthermore since

$$\dot{\beta}_1(t) = \left(0, \dots, 0, \frac{\beta(t) \cdot \dot{\beta}(t)}{|\beta(t)|}\right),$$

we have

$$|\dot{\beta}_1(t)| = \frac{|\beta(t) \cdot \dot{\beta}(t)|}{|\beta(t)|} \leq |\dot{\beta}(t)|$$

for all $t \in [a, b]$ (by the Cauchy–Schwartz inequality). It follows that

$$\begin{aligned}
|\beta| &= \int_a^b \langle \dot{\beta}(t), \dot{\beta}(t) \rangle_{\beta(t)}^{1/2} dt \\
&= \int_a^b \left( \frac{(1 - |\beta(t)|)|\dot{\beta}(t)|^2 + (\beta(t) \cdot \dot{\beta}(t))^2}{(1 - |\beta(t)|^2)^2} \right)^{1/2} dt \\
&\geq \int_a^b \left( \frac{(1 - |\beta_1(t)|)|\dot{\beta}_1(t)|^2 + (\beta_1(t) \cdot \dot{\beta}_1(t))^2}{(1 - |\beta_1(t)|^2)^2} \right)^{1/2} dt \\
&= \int_a^b \langle \dot{\beta}_1(t), \dot{\beta}_1(t) \rangle_{\beta_1(t)}^{1/2} dt \\
&= |\beta_1|.
\end{aligned}$$

Since $\beta_1$ is also a curve such that $\beta_1(a) = y$ and $\beta_1(b) = x$, the result follows. $\square$

The proof of Lemma 1.2 is now completed by noting that $\nu$ maps the points $x$ and $y$ defined in the proof of Lemma 1.4 above to two points of the required form, $\nu$ preserves geodesics and that $\nu$ maps the line segment $\{(0, \dots, 0, a) : a \in (-1, 1)\}$ onto the intersection with $H^n$ of the plane $\{(a, 0, \dots, 0, b) : a, b \in \mathbf{R}\}$ of $\mathbf{R}^{n+1}$.

We now use $\mu$ to transfer geodesics and the distance function to $B^n$.

**Theorem 1.5** *The geodesics of $B^n$ with the Klein metric are the intersections of Euclidean straight lines with $B^n$. The associated distance function is given by*

$$d(x,y) = \cosh^{-1}\left(\frac{1 - x \cdot y}{\sqrt{1-|x|^2}\,\sqrt{1-|y|^2}}\right).$$

**Proof** By Theorem 1.3, the geodesics of $H^n$ are the intersections with $H^n$ of planes through the origin of $\mathbf{R}^{n+1}$. The gnomonic projection $\mu$ maps points on a plane through the origin of $\mathbf{R}^{n+1}$ onto the same plane, hence the geodesics of $B^n$, embedded in $\{1\} \times \mathbf{R}^n$ as discussed above, are indeed Euclidean line segments. Now let $x, y \in B^n$. Since $\nu$ is an isometry, we have

$$\begin{aligned} d(x,y) &= d(\nu(x), \nu(y)) \\ &= \cosh^{-1}\left(-\langle \nu(x), \nu(y)\rangle\right) \\ &= \cosh^{-1}\left(-\left\langle \frac{(1,x)}{\sqrt{1-|x|^2}}, \frac{(1,y)}{\sqrt{1-|y|^2}}\right\rangle\right) \\ &= \cosh^{-1}\left(\frac{1 - x \cdot y}{\sqrt{1-|x|^2}\,\sqrt{1-|y|^2}}\right) \end{aligned}$$

as claimed. □

### 1.2.2 The Conformal Ball Model

The next model that we examine is the conformal ball model, sometimes known as the Poincaré model. This model also uses $B^n$ as its underlying space, but the new Riemannian metric differs from the Klein metric. The angle between two tangent vectors arising from the new Riemannian metric agrees with the Euclidean angle between these vectors, however the geodesics are no longer straight line segments but are in fact arcs of circles which intersect the boundary $S^{n-1}$ at right angles.

**Definition** The *(hyperbolic) stereographic projection* $\zeta : H^n \longrightarrow B^n$ is defined by

$$\zeta(x_0, x_1, \ldots, x_n) = \left(\frac{x_1}{1+x_0}, \ldots, \frac{x_n}{1+x_0}\right)$$

for all $x = (x_0, \ldots, x_n) \in H^n$. Note that $\zeta$ is well defined, since $x_0^2 - x_1^2 - \cdots - x_n^2 = 1$ implies that $x_1^2 + \cdots + x_n^2 = x_0^2 - 1 < (1+x_0)^2$. It is also a bijection (since $x_0 > 0$), with inverse given by

$$\zeta^{-1}(y_1, \dots, y_n) = \left( \frac{1+|y|^2}{1-|y|^2}, \frac{2y_1}{1-|y|^2}, \dots, \frac{2y_n}{1-|y|^2} \right)$$

for all $y = (y_1, \dots, y_n) \in B^n$. Geometrically, $\zeta(x)$ is the intersection of the line joining $-e_0$ to $x$ with the $n$-dimensional subspace orthogonal to $e_0$. The unit ball $B^n$ may be thought of as being embedded in $\mathbf{R}^{n+1}$ by identifying $y \in B^n$ with the point $(0, y) \in \mathbf{R}^{n+1}$.

As in the analysis of the Klein model, we use $\zeta^{-1}$ to transfer the Riemannian metric of $H^n$ to $B^n$ by requiring that $\zeta$ be an isometry. A simple calculation shows that the Jacobian of $\zeta^{-1}$ at the point $x = (x_1, \dots, x_n) \in B^n$ is given by

$$J = \frac{1}{\lambda^4} \begin{pmatrix} 4x_1 & 4x_2 & \cdots & 4x_n \\ 2\lambda^2 + 4x_1^2 & 4x_1x_2 & \cdots & 4x_1x_n \\ 4x_2x_1 & 2\lambda^2 + 4x_2^2 & \cdots & 4x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ 4x_nx_1 & 4x_nx_2 & \cdots & 2\lambda^2 + 4x_n^2 \end{pmatrix}$$

where $\lambda = \sqrt{1 - x_1^2 - \cdots - x_n^2} = \sqrt{1 - |x|^2}$ as before. Then if $y \in T_xB^n$, we see that

$$Jy = \frac{4(x \cdot y)}{\lambda^4}(1, x) + \frac{2}{\lambda^2}(0, y)$$

and that

$$\langle Jy, Jy \rangle = \frac{4|y|^2}{\lambda^4}.$$

If we transfer the metric using the obvious formula

$$\langle y, y \rangle_x = \langle \dot{\xi}(y), \dot{\xi}(y) \rangle_{\xi(x)},$$

where $\xi = \zeta^{-1}$, then we obtain

$$\langle y, y \rangle_x = \frac{4|y|^2}{(1-|x|^2)^2}.$$

We call this metric the *Poincaré metric* on $B^n$.

Recall that a Möbius transformation of $\mathbf{R}^n$ is a map which is expressible as a composition of reflections in spheres and hyperplanes.

**Theorem 1.6** *The isometries of $B^n$ with the Poincaré metric are precisely the restrictions to $B^n$ of Möbius transformations which preserve $B^n$.*

**Proof** It is not difficult to check that every Möbius transformation of $B^n$ (that is, a Möbius transformation which preserves $B^n$) restricts to an isometry of $B^n$ with the Poincaré metric (see pp. 128–129 of [R] for the calculation). To prove the converse, we use $\zeta$ and $\xi$ to transfer isometries of $H^n$ to isometries of $B^n$. For each $b \in (-1,1)$ with $b \neq 0$, define the map $\tau_b : B^n \to B^n$ by

$$\tau_b(y) = \left(b^2|y|^2 + 2y_n b + 1\right)^{-1} \left(\left(1-b^2\right) y + \left(|y|^2 + 2y_n b + 1\right) b e_n\right)$$

for all $y = (y_1, \ldots, y_n) \in B^n$. In fact $\tau_b$ is the restriction to $B^n$ of $\rho_b \sigma_b$, where $\rho_b$ is the reflection in the $(n-1)$-dimensional subspace orthogonal to $e_n$, and $\sigma_b$ is the reflection in the sphere with centre $-b^{-1}e_n$ and radius $|b|^{-1}\sqrt{1-b^2}$. Since $\rho_b$ and $\sigma_b$ are both Möbius transformations of $B^n$, $\tau_b$ is also a Möbius transformation of $B^n$. A simple calculation shows that

$$\tau_b = \zeta a(t) \zeta^{-1},$$

where

$$a(t) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_n & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \in A$$

and

$$b = \frac{\sinh t}{1 + \cosh t}.$$

(Here we refer to the Cartan decomposition $I(H^n) = KAK$.) Now if $\psi$ is any rotation of $B^n$, then $\psi$ is the restriction to $B^n$ of an orthogonal transformation $R \in O(n)$. Such a transformation is evidently a Möbius transformation of $B^n$ (by composing at most $(n+1)$ reflections in hyperplanes; see p. 106 of [R]). Furthermore

$$\psi = \zeta k \zeta^{-1},$$

where

$$k = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \in K.$$

Since $I(H^n) = KAK$ and we have demonstrated that $A$ and $K$ act on $B^n$ as the restriction to $B^n$ of Möbius transformations of $B^n$ (using $\zeta$), we conclude that every isometry of $B^n$ with the Poincaré metric is indeed the restriction to $B^n$ of a Möbius transformation of $B^n$. $\square$

It is evident that $\zeta$ maps geodesics through $(1, 0, \dots, 0) \in H^n$ to diameters of $B^n$. These diameters are arcs of (degenerate) circles which intersect $S^{n-1}$ at right angles. It is well known that the images of lines under Möbius transformations are circles or lines. It follows by conformality that any Möbius transformation of $B^n$ maps the diameters of $B^n$ into arcs of circles orthogonal to $S^{n-1}$.

**Theorem 1.7** *The geodesics of $B^n$ with the Poincaré metric are arcs of circles intersecting $S^{n-1}$ at right angles. The associated distance function is given by*

$$d(x, y) = \cosh^{-1}\left(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\right)$$

*for all $x, y \in B^n$.*

**Proof** The geodesics have already been identified. To find the distance formula, we use the fact that $\xi$ is an isometry, obtaining

$$\begin{aligned}
d(x, y) &= d(\xi(x), \xi(y)) \\
&= \cosh^{-1}\left(-\langle \xi(x), \xi(y) \rangle\right) \\
&= \cosh^{-1}\left(\frac{(1+|x|^2)(1+|y|^2)}{(1-|x|^2)(1-|y|^2)} - \sum_{k=1}^{n} \frac{4x_k y_k}{(1-|x|^2)(1-|y|^2)}\right) \\
&= \cosh^{-1}\left(\frac{(1-|x|^2)(1-|y|^2) + 2(|x|^2+|y|^2) - 4(x \cdot y)}{(1-|x|^2)(1-|y|^2)}\right) \\
&= \cosh^{-1}\left(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\right)
\end{aligned}$$

for all $x, y \in B^n$, as claimed. $\square$

Let $y, z \in T_x B^n$. By polarisation, we see that

$$\langle y, z \rangle_x = \frac{4(y \cdot z)}{(1-|x|^2)^2},$$

thus we have

$$\frac{\langle y, z \rangle_x}{\langle y, y \rangle_x^{1/2} \langle z, z \rangle_x^{1/2}} = \frac{\dfrac{4(y \cdot z)}{(1-|x|^2)^2}}{\dfrac{2|y|}{1-|x|^2} \dfrac{2|z|}{1-|x|^2}} = \frac{y \cdot z}{|y|\,|z|}.$$

If we interpret the left hand side as the hyperbolic (Poincaré) angle between $y$ and $z$, then this equation shows that the Poincaré angle agrees with the Euclidean angle, regardless of the choice of $x \in B^n$. This justifies the name "conformal ball model".

### 1.2.3 The Upper Half-Space Model

The final model that we examine is the upper half-space model. The upper half-space $U^n = \{(x_0, \ldots, x_{n-1}) \in \mathbf{R}^n : x_0 > 0\}$ is the base space for this model, which is essentially the Poincaré model transferred to $U^n$ using a Möbius transformation. This model has the advantage that the metric has a particularly simple form.

Let $\eta : B^n \to U^n$ denote the Möbius transformation from $B^n$ to $U^n$ defined by

$$x = (x_0, x_1, \ldots, x_{n-1}) \mapsto \frac{1}{1 - 2x_0 + |x|^2} \left(1 - |x|^2, 2x_1, \ldots, 2x_{n-1}\right).$$

Treating $\eta$ as an isometry, we may transfer the Poincaré metric from $B^n$ to $U^n$ by defining

$$\langle y, y \rangle_x = \langle \dot{\tau}_x(y), \dot{\tau}_x(y) \rangle_{\tau(x)}$$

for all $x \in U^n$ and all $y \in T_x U^n$, where $\tau = \eta^{-1}$.

**Theorem 1.8** *The Poincaré metric on $U^n$ is given by*

$$\langle y, y \rangle_x = \frac{|y|^2}{x_0^2}$$

*for all $x = (x_0, \ldots, x_{n-1}) \in U^n$ and all $y \in T_x U^n$. The associated distance formula is given by*

$$d(x, y) = \cosh^{-1} \left(1 + \frac{|x - y|^2}{2x_0 y_0}\right)$$

*for all $x = (x_0, \ldots, x_{n-1}), y = (y_0, \ldots, y_{n-1}) \in U^n$. The geodesics are circular arcs and rays intersecting the boundary $\mathbf{R}^{n-1} = \{(0, x_1, \ldots, x_{n-1}) : x_1, \ldots, x_{n-1} \in \mathbf{R}\}$ orthogonally.*

The calculations involved in the proof of this theorem are straightforward and may be found on pp. 136–139 of [R]. Note that this metric is also conformal.

## 1.3 Complex Hyperbolic Geometry

The construction of complex hyperbolic geometry proceeds along similar lines to the real case, however there are intrinsic differences which arise from the properties of complex numbers. Complex hyperbolic $n$-space is not isomorphic to real hyperbolic $2n$-space for $n > 1$, for in the definition of the Lorentzian form (1.1), one particular direction is distinguished from the others by virtue of the minus sign in the expression. On the other hand, in the complex Lorentzian form given below (1.3), one *complex* direction is distinguished. This direction corresponds to *two* real directions. The most obvious indication that the projective disc models of complex hyperbolic $n$-space and real hyperbolic $2n$-space are different is that the geodesics in the latter space are straight lines whereas the geodesics in the former space are arcs of circles. The material in this section, particularly the proof of Lemma 1.9, is based on [E].

**Definition** The *Lorentzian form* $\langle \cdot , \cdot \rangle$ on $\mathbf{C}^{n+1}$ is given by

$$\langle z, w\rangle = -\overline{z_0}w_0 + \overline{z_1}w_1 + \cdots + \overline{z_n}w_n \tag{1.3}$$

for all $z = (z_0, \ldots , z_n)$, $w = (w_0, \ldots , w_n) \in \mathbf{C}^{n+1}$. Define complex hyperbolic space $H^n(\mathbf{C})$ by

$$H^n(\mathbf{C}) = \{[z] \in P^{n+1}(\mathbf{C}) : \langle z, z\rangle < 0\}.$$

Here $P^{n+1}(\mathbf{C})$ denotes projective complex $(n + 1)$-space. The class of $z$, denoted by $[z]$, is the equivalence class containing $z$ under the equivalence relation $\sim$, where $z \sim w$ if and only if there exists $\lambda \in \mathbf{C}^* = \mathbf{C}\setminus\{0\}$ with $z = \lambda w$. Note that $H^n(\mathbf{C})$ is well-defined, for if $z \in \mathbf{C}^{n+1}$ such that $\langle z, z\rangle < 0$ and $\lambda \in \mathbf{C}^*$, then $\langle \lambda z, \lambda z\rangle = |\lambda|^2\langle z, z\rangle < 0$.

The following result is the analogue of the Cauchy–Schwartz inequality for complex vector spaces.

**Lemma 1.9** *Let $z, w \in \mathbf{C}^{n+1}$ such that $\langle z, z\rangle \leq 0$ and $\langle w, w\rangle \leq 0$. Then*

$$\langle z, z\rangle\langle w, w\rangle \leq \langle z, w\rangle\langle w, z\rangle.$$

**Proof** If $\langle w, w\rangle = 0$, the result is trivial, so assume $\langle w, w\rangle < 0$. We claim that the set

$$S = \{\lambda \in \mathbf{C} : \langle z + \lambda w, z + \lambda w\rangle \geq 0\}$$

is compact. It is certainly closed, and if $\lambda \in S$, then

$$2\left|\lambda\langle w,z\rangle\right| \geq 2\mathrm{Re}\left(\lambda\langle w,z\rangle\right) \geq -\langle z,z\rangle - |\lambda|^2\langle w,w\rangle,$$

so

$$-|\lambda|^2\langle w,w\rangle - 2|\lambda|\left|\langle w,z\rangle\right| - \langle z,z\rangle \leq 0$$

which ensures that $|\lambda|$ is bounded above. Furthermore, $S$ is nonempty, for we must have $w_0 \neq 0$ (or else $\langle w,w\rangle \geq 0$), so if $\lambda = -z_0/w_0$, then

$$\langle z+\lambda w, z+\lambda w\rangle = -|z_0+\lambda w_0|^2 + \sum_{i=1}^{n} |z_i+\lambda w_i|^2 = \sum_{i=1}^{n} |z_i+\lambda w_i|^2 \geq 0,$$

that is, $\lambda \in S$. The function $f : \lambda \mapsto \langle z+\lambda w, z+\lambda w\rangle$ is continuous on $S$, thus it attains its maximum at some $\lambda_0 \in \mathbf{C}$. Considering $f$ as a function of two real variables, we have

$$f(x,y) = (x^2+y^2)\langle w,w\rangle + 2(\alpha x - \beta y) + \langle z,z\rangle$$

where $\langle w,z\rangle = \alpha + i\beta$ for some $\alpha$ and $\beta$ in $\mathbf{R}$. Setting $\lambda_0 = x_0 + iy_0$, we have $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$, so

$$\begin{aligned} 2x_0\langle w,w\rangle + 2\alpha &= 0 \\ 2y_0\langle w,w\rangle - 2\beta &= 0. \end{aligned}$$

Solving for $x_0, y_0$,

$$\begin{aligned} x_0 &= -\frac{\mathrm{Re}\langle w,z\rangle}{\langle w,w\rangle} \\ y_0 &= \frac{\mathrm{Im}\langle w,z\rangle}{\langle w,w\rangle}. \end{aligned}$$

That is,

$$\lambda_0 = -\frac{\langle z,w\rangle}{\langle w,w\rangle},$$

thus we have

$$\left\langle z - \frac{\langle z,w\rangle}{\langle w,w\rangle}w, z - \frac{\langle z,w\rangle}{\langle w,w\rangle}w \right\rangle \geq 0.$$

Multiplying through by $-\langle w,w\rangle$, expanding and rearranging gives the result. □

Let $z \in \mathbf{C}^{n+1}$ with $\langle z, z \rangle < 0$. The tangent space to $[z] \in H^n(\mathbf{C})$ may be identified with $z^\perp = \{w \in \mathbf{C}^{n+1} : \langle z, w \rangle = 0\}$. Here $z^\perp$ is itself identified with $(\lambda z)^\perp$ by multiplication by $\lambda \in \mathbf{C}^*$. That is, if $w$ is a tangent vector at $z$, then $\lambda w$ is the equivalent tangent vector at the equivalent point $\lambda z$, $\lambda \in \mathbf{C}^*$. Then if $z, w \in \mathbf{C}^{n+1}$, we may define

$$\langle w, w \rangle_z = \frac{\langle z, z \rangle \langle w, w \rangle - \langle z, w \rangle \langle w, z \rangle}{-\langle z, z \rangle^2}$$

to be the inner product of $w$ with itself at the point $z$. Lemma 1.9 implies that this inner product is positive definite on the tangent space at each point of $H^n(\mathbf{C})$.

We now define the complex analogue of the Lorentz group by

$$\begin{aligned} U(1,n) &= \{A \in GL(n+1, \mathbf{C}) : \langle Az, Aw \rangle = \langle z, w \rangle \text{ for all } z, w \in \mathbf{C}^{n+1}\} \\ &= \{A \in GL(n+1, \mathbf{C}) : A^* S A = S\} \end{aligned}$$

where

$$S = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.$$

Define an equivalence relation on $U(1,n)$ by $A_1 \sim A_2$ if and only if there exists $\lambda \in \mathbf{C}, |\lambda| = 1$ such that $A_1 = \lambda A_2$. Let $PU(1,n)$ denote the corresponding factor group $U(1,n)/\sim$. Then the group $I(H^n(\mathbf{C}))$ of isometries of complex hyperbolic $n$-space is given by

$$I(H^n(\mathbf{C})) = PU(1,n) \cup \sigma\, PU(1,n),$$

where $A \in PU(1,n)$ acts on $[z] \in H^n(\mathbf{C})$ by $A([z]) = [Az]$ and $\sigma$ is componentwise complex conjugation $\sigma([z]) = [\overline{z}]$. As in the real case, we have the Cartan decomposition $U(1,n) = KAK$, where (up to equivalence)

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} : R \in U(n) \right\} \cup \left\{ \sigma \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} : R \in U(n) \right\}$$

and

$$A = \left\{ a(t) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \cosh t & 0 & \sinh t \end{pmatrix} : t \in \mathbf{R} \right\}.$$

This implies that $U(1,n)$ acts transitively on $H^n(\mathbf{C})$. The proof is identical to the proof of the real case given in Section 1.1.

Given $[z] \in H^n(\mathbf{C})$, we note that $z_0 \neq 0$, or else we would have $\langle z, z\rangle \geq 0$. If we set $\lambda = z_0^{-1} \in \mathbf{C}^*$, then $(\lambda z)_0 = 1$. That is, we can always choose a (unique) representative of $[z]$ such that $z_0 = 1$. The condition that $\langle z, z\rangle < 0$ implies that $|z_1|^2 + \cdots + |z_n|^2 < 1$, that is, $(z_1, z_2, \ldots, z_n) \in B^n(\mathbf{C})$, the unit ball in $\mathbf{C}^n$. If $w \in T_z B^n(\mathbf{C})$, we may regard $z$ as the point $(1, z_1, \ldots, z_n) \in \mathbf{C}^{n+1}$ and $w$ as the point $(0, w_1, \ldots, w_n) \in \mathbf{C}^{n+1}$; we have

$$\langle w, w\rangle_z = \frac{(-1+|z|^2)|w|^2 - |(z,w)|^2}{-(-1+|z|^2)^2}$$

where $|v|^2 = |v_1|^2 + \cdots + |v_n|^2$ and $(u,v) = \overline{u_1}v_1 + \cdots + \overline{u_n}v_n$ for any $u, v \in \mathbf{C}^n$. In particular, the formula for the element of arc length is given by setting $w = dz$, obtaining

$$ds^2 = \frac{(1-|z|^2)|dz|^2 + |(z,dz)|^2}{(1-|z|^2)^2}.$$

The space $B^n(\mathbf{C})$ with this metric is referred to as the *(complex) projective disc model.* It is the complex analogue of the Klein model of real hyperbolic geometry.

In order to find the geodesics of $H^n(\mathbf{C})$, we first establish a connection between real and complex hyperbolic geometry.

**Lemma 1.10** *Suppose $n \geq 2$. Let $[z] \in H^n(\mathbf{C})$ and $[w] \in P^{n+1}(\mathbf{C})\backslash\{0\}$. Then the intersection of $H^n(\mathbf{C})$ with the projective complex line $L = \{[\lambda z + \mu w] : \lambda, \mu \in \mathbf{C}\}$ may be identified with $H^2$. In particular, the (non-empty) intersection of $B^n(\mathbf{C})$ with any complex line is isomorphic to $B^2$ with one-quarter of the (real) Poincaré metric.*

**Proof** Since $z_0$ is nonzero, we may choose $z$ such that $z_0 = 1$. We may replace $w$ with $w - w_0 z$ without affecting $L$; then $w_0 = 0$. Passing to the projective disc model, we now relabel $z = (1, z)$ and $w = (0, w)$ with $z \in B^n(\mathbf{C})$ and $w \in \mathbf{C}^n\backslash\{0\}$. We may replace $w$ and $z$ with $(w,w)^{-1/2}w$ and $z - (w,w)^{-1}(z,w)w$ respectively and thereby insist without loss of generality that $|w| = 1$ and $(z, w) = 0$. Since the metric on $B^n(\mathbf{C})$ is invariant under unitary transformations, we may choose an orthonormal basis such that $z = (0, t, 0, \ldots, 0)$ and $w = (1, 0, \ldots, 0)$, where $0 \leq t < 1$, so that

$L$ is represented by the complex line $\{(\lambda, t, 0, \dots, 0) : \lambda \in \mathbf{C}\}$. Let $p \in L$ with $p = (\lambda, t, 0, \dots, 0\}$, $\lambda \in \mathbf{C}$, and let $v \in T_p(L \cap B^n(\mathbf{C})$; then $v = (\mu, 0, \dots, 0)$ for some $\mu \in \mathbf{C}$. We have

$$\langle v, v \rangle_p = \frac{(1 - |\lambda|^2 - t^2)|\mu|^2 + |\lambda\overline{\mu}|^2}{(1 - |\lambda|^2 - t^2)^2} = \frac{|\mu|^2(1 - t^2)}{(1 - t^2 - |\lambda|^2)^2}.$$

Define $f : \mathbf{C} \to \mathbf{R}^2$ and $F : L \to \mathbf{R}^2$ by

$$f(\lambda) = (\mathrm{Re}(\lambda), \mathrm{Im}(\lambda)) \qquad \text{and} \qquad F(\lambda, t, 0, \dots, 0) = f(\lambda)$$

for all $\lambda \in \mathbf{C}$. If $x \in \mathbf{R}^2$, set $p = F^{-1}(x)$ and $v = (f^{-1}(dx), 0, \dots, 0)$ to obtain

$$ds^2 = \frac{r^2|dx|^2}{(r^2 - |x|^2)^2} \tag{1.4}$$

where $r = \sqrt{1 - t^2}$. We have shown that $L \cap B^n(\mathbf{C})$ is isometrically isomorphic (using $F$) to $B(0, r) \subset \mathbf{R}^2$ with the above metric (1.4). The transformation $g : x \mapsto x/r$ of $\mathbf{R}^2$ maps $B(0, r)$ isomorphically onto $B(0, 1)$ and transforms (1.4) into

$$ds^2 = \frac{|dx|^2}{(1 - |x|^2)^2}$$

which is one-quarter of the Poincaré metric on $B(0, 1) \subset \mathbf{R}^2$. □

Lemma 1.10 allows us to describe the geodesics in complex hyperbolic space.

**Theorem 1.11** *If $z, w \in B^n(\mathbf{C})$, $z \neq w$, then there is a unique geodesic containing $z$ and $w$. It lies in the unique complex line $L$ containing $z$ and $w$. Furthermore, if $z$, $w$, $B^n(\mathbf{C})$ and $L$ are identified with $\tilde{z} \in \mathbf{R}^{2n}$, $\tilde{w} \in \mathbf{R}^{2n}$, $B^{2n}$ and a real plane $\pi \subset \mathbf{R}^{2n}$ respectively under the natural identification of $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, then the geodesic joining $z$ and $w$ is identified with the unique circular arc in $B^{2n}$ through $\tilde{z}$ and $\tilde{w}$ which lies in the 2-disc $\pi \cap B^{2n}$ and intersects the boundary of this disc at right-angles.*

**Proof** We have seen that $U(1, n)$ acts transitively on $B^n(\mathbf{C})$ and maps geodesics to geodesics. We may therefore choose $A \in U(1, n)$ such that $A(z) = 0$. By an argument similar to the one used in the proof of Lemma 1.4, the intersection of the real line $\{tA(w) : t \in \mathbf{R}\}$ with $B^n(\mathbf{C})$ is the unique geodesic containing 0 and $A(w)$. This geodesic evidently lies in the complex line $\{\lambda A(w) : \lambda \in \mathbf{C}\}$. Now $A^{-1}$ maps the corresponding projective line $\{[\lambda A(1, w)] : \lambda \in \mathbf{C}\}$ into the complex projective line $L'$ through $[(1, z)]$ and $[(1, w)]$. If representatives $v$ of points $[v]$ on $L'$ are chosen to have $v_0 = 1$, then $L'$ may be represented in $B^n(\mathbf{C})$ as the complex line $L$ containing $z$ and $w$. This proves the first assertion. By Lemma 1.10, $L \cap B^n(\mathbf{C})$ is isometrically isomorphic to a dilation of $B^2$ with metric equal to one-quarter of the Poincaré metric. As seen in Section 1.2.2, the geodesics in this model are arcs of circles which intersect the boundary of $B^2$ at right-angles; this remains the case even if the metric is multiplied by $1/4$ (although the lengths of geodesic arcs are different). □

We conclude this chapter by deriving the distance function on $H^n(\mathbf{C})$.

**Theorem 1.12** *Let* $[z], [w] \in H^n(\mathbf{C})$. *Then*

$$d([z],[w]) = \cosh^{-1} \sqrt{\frac{\langle z, w\rangle \langle w, z\rangle}{\langle z, z\rangle \langle w, w\rangle}}.$$

*Equivalently, if* $z, w \in B^n(\mathbf{C})$, *then*

$$d(z, w) = \cosh^{-1} \frac{|1 - (z, w)|}{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}.$$

**Proof** As noted previously, $U(1, n)$ acts transitively on $H^n(\mathbf{C})$ and preserves distances. Choose $A \in U(1, n)$ such that $A([z]) = [(1, 0, \dots, 0)]$ and let $[v] = A([w])$. Then by Theorem 1.11, if $v$ is the representative of $[v]$ such that $v_0 = 1$, then the geodesic arc from $[(1, 0, \dots, 0)]$ to $[v]$ is given by

$$L = \{[(1, tv_1, \dots, tv_n)] : t \in [0, 1]\}.$$

This is represented in $B^n(\mathbf{C})$ by the curve $\gamma : [0, 1] \to B^n(\mathbf{C})$ given by

$$\gamma(t) = (tv_1, \dots, tv_n).$$

The length of this curve is given by

$$\begin{aligned}
|\gamma| &= \int_0^1 \langle \dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)}^{1/2} dt = \int_0^1 \langle v, v\rangle_{(tv_1, \dots, tv_n)}^{1/2} dt \\
&= \int_0^1 \sqrt{\frac{(1 - t^2|v|^2)|v|^2 + |(tv, v)|^2}{(1 - t^2|v|^2)^2}} dt = \int_0^1 \frac{|v|}{1 - t^2|v|^2} dt \\
&= \frac{1}{2} \log\left(\frac{1 - |v|}{1 + |v|}\right) = \cosh^{-1}\left(\frac{1}{\sqrt{1 - |v|^2}}\right) \\
&= \cosh^{-1} \sqrt{\frac{\langle (1, 0, \dots, 0), v\rangle \langle v, (1, 0, \dots, 0)\rangle}{\langle (1, 0, \dots, 0), (1, 0, \dots, 0)\rangle \langle v, v\rangle}}.
\end{aligned}$$

(Here we have abused notation slightly: the last line refers to $v = (1, v_1, \dots, v_n)$ as an element of $\mathbf{C}^{n+1}$ whereas the other lines treat $v = (v_1, \dots, v_n)$ as being in $B^n(\mathbf{C})$.) It follows that

$$\begin{aligned}
d([z],[w]) &= d([Az],[Aw]) \\
&= \cosh^{-1}\sqrt{\frac{\langle Az, Aw\rangle \langle Aw, Az\rangle}{\langle Az, Az\rangle \langle Aw, Aw\rangle}} \\
&= \cosh^{-1}\sqrt{\frac{\langle z, w\rangle \langle w, z\rangle}{\langle z, z\rangle \langle w, w\rangle}}
\end{aligned}$$

as required. Note that this formula is well-defined, that is, it is independent of the representatives of $[z]$ and $[w]$. The equivalent expression for $z, w \in B^n(\mathbf{C})$ follows easily from the above formula. $\square$

# Chapter 2

# Clifford Algebras, Spin Groups and Octonions

Our construction of symmetric spaces of rank one of noncompact type utilises Clifford algebras and the associated so-called Spin groups. In this chapter we describe these objects and explore the relationship between certain Clifford algebras and the division algebras of quaternions and octonions. The emphasis is on the aspects of Clifford theory which relate to the construction of symmetric spaces. For a more general approach, see Porteous [Ps], from which much of the material in this chapter is derived.

## 2.1 Quaternions

The algebra $\mathbf{H}$ of *quaternions* is the space $\mathbf{R}^4$ with product defined by

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik,$$

where $\{1, i, j, k\}$ is the standard (orthonormal) basis for $\mathbf{R}^4$. This product is associative and respects the norm $|\cdot| : \mathbf{H} \to \mathbf{R}$ given by

$$|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

for all $a, b, c, d \in \mathbf{R}$, that is, $|xy| = |x|\,|y|$ for all $x, y \in \mathbf{H}$. This implies that $\mathbf{H}$ is a division algebra, with

$$x^{-1} = |x|^{-2}\,\overline{x},$$

where $\overline{x} = a - bi - cj - dk$ for all $x = a + bi + cj + dk \in \mathbf{H}$. It also implies that the 3-sphere

$$S^3 = \{x \in \mathbf{H} : |x| = 1\}$$

is a group under quaternionic multiplication. For $x = a + bi + cj + dk \in \mathbf{H}$, define $\mathrm{Re}(x) = a$ and $\mathrm{Im}(x) = bi + cj + dk$, so that

$$\mathrm{Re}(x) = \frac{x + \bar{x}}{2}, \quad \mathrm{Im}(x) = \frac{x - \bar{x}}{2}.$$

We call $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ the *real* and *imaginary parts* of $x$ respectively. The set of imaginary quaternions $\mathrm{Im}(\mathbf{H}) = \{h \in \mathbf{H} : h = \mathrm{Im}(h)\}$ is clearly isomorphic to $\mathbf{R}^3$. (Note that we treat 0 as both real and imaginary.) When there is no ambiguity, the space $\mathbf{R}^3$ will denote the imaginary quaternions. It is easy to see that

$$\mathbf{R}^3 = \{h \in \mathbf{H} : h^2 \leq 0\}.$$

The map $x \mapsto \bar{x}$ is referred to as *conjugation* and satisfies

$$\overline{xy} = \overline{y}\,\overline{x}$$

for all $x, y \in \mathbf{H}$. We also have

$$x \cdot y = \mathrm{Re}(\overline{x}y), \qquad w \times z = \mathrm{Im}(wz)$$

for all $x, y \in \mathbf{H}$, $w, z \in \mathbf{R}^3$, where the dot product is taken in $\mathbf{R}^4$ and the cross product is taken in $\mathbf{R}^3$. In particular, two imaginary quaternions $w$ and $z$ anticommute if and only if they are orthogonal.

**Lemma 2.1** *For $q \in \mathbf{H}^* = \mathbf{H}\backslash\{0\}$, $x \in \mathbf{R}^3$, the quaternion $qxq^{-1}$ is imaginary; furthermore if $q \in \mathbf{R}^3\backslash\{0\}$, the map*

$$\rho_q : \mathbf{R}^3 \to \mathbf{R}^3; \quad x \mapsto -qxq^{-1}$$

*is the reflection in $\{\mathbf{R}q\}^\perp \subseteq \mathbf{R}^3$, the plane through 0 with normal $q$.*

**Proof** Since $x^2$ is real and nonpositive,

$$(\rho_q x)^2 = \left(-qxq^{-1}\right)^2 = qx^2q^{-1} = x^2qq^{-1} = x^2 \leq 0,$$

hence $\rho_q x \in \mathbf{R}^3$. Furthermore $\rho_q q = -q$, whereas if $r \in \mathbf{R}^3$ is orthogonal to $q$ we have

$$\rho_q r = -qrq^{-1} = rqq^{-1} = r$$

since $r$ and $q$ anti-commute. The result follows by the linearity of $\rho_q$. □

**Corollary** *A map $g \in SO(3)$ if and only if it is of the form $-\rho_q$ for some $q \in \mathbf{H}^*$.*

**Proof** This follows from the fact that any element of $SO(3)$ is generated by two reflections in planes through the origin and $\rho_q \rho_r = -\rho_{qr}$ for all $q, r \in \mathbf{R}^3 \backslash \{0\}$. $\square$

**Definition** If $A$ is an algebra then a map $u : A \to A$ is an *automorphism* (*anti-automorphism*) if it is linear and $u(ab) = u(a)u(b)$ (respectively $u(ab) = u(b)u(a)$) for all $a, b \in A$. A map $u : A \to A$ is an *involution* (*anti-involution*) if it is an automorphism (respectively anti-automorphism) satisfying $u^2 = I$.

**Lemma 2.2** *A map $u : \mathbf{H} \to \mathbf{H}$ is an automorphism (anti-automorphism) if and only if there exists $R \in SO(3)$ (respectively $R \in O(3) \backslash SO(3)$) such that*

$$u(x) = \mathrm{Re}(x) + R(\mathrm{Im}(x))$$

*for all $x \in \mathbf{H}$.*

**Proof** If $u(x) = \mathrm{Re}(x) + R(\mathrm{Im}(x))$ for $R \in SO(3)$, then we may write $R = -\rho_q$ for some $q \in \mathbf{H}^*$, so

$$u(x) = qxq^{-1}$$

which is an automorphism of $\mathbf{H}$. Alternatively if $u(x) = \mathrm{Re}(x) + R(\mathrm{Im}(x))$ for $R \in O(3) \backslash SO(3)$, then $-R \in SO(3)$, and we have

$$u(x) = \overline{\mathrm{Re}(x) - R(\mathrm{Im}(x))} = \overline{qxq^{-1}}$$

for some $q \in \mathbf{H}^*$, implying that $u$ is an anti-automorphism.

Conversely, suppose that $u$ is an automorphism or anti-automorphism of $\mathbf{H}$. Since $u(a) = u(1)u(a)$ for all $a \in \mathbf{H}$, we must have $u(1) = 1$. Furthermore, if $x \in \mathbf{R}^3$ then $u(x)^2 = u(x^2) = x^2 \leq 0$, so $u(x) \in \mathbf{R}^3$ and $|u(x)| = |x|$. It follows that $u$ is of the required form with $R \in O(3)$ equal to the restriction of $u$ to $\mathbf{R}^3$. By the above remarks, $u$ is an automorphism if $R \in SO(3)$ and an anti-automorphism otherwise. $\square$

**Corollary** *A map $u : \mathbf{H} \to \mathbf{H}$ is an involution if and only if it is the identity or it corresponds to the rotation of $\mathbf{R}^3$ through $\pi$ about some line through the origin. The map $u$ is an anti-involution if and only if it is the composition of an involution with conjugation, that is, if and only if it corresponds to the reflection of $\mathbf{R}^3$ in the origin or the reflection of $\mathbf{R}^3$ in some plane through the origin.*

We call the involution $\widehat{\cdot}: \mathbf{H} \to \mathbf{H}; x \mapsto \widehat{x} = jxj^{-1} = -jxj$ the *main involution* and its associated anti-involution $\tilde{\cdot}: \mathbf{H} \to \mathbf{H}; x \mapsto \tilde{x} = \widehat{\overline{x}} = \overline{\widehat{x}}$ the *reversion anti-involution.*

## 2.2 Tensor Products of Algebras

For $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, let $\mathbf{K}(n)$ denote the real algebra of $n \times n$ matrices with entries in $\mathbf{K}$, and let ${}^2\mathbf{K}(n)$ denote the space $(\mathbf{K}(n)) \oplus (\mathbf{K}(n))$ endowed with the product

$$(a, b) \cdot (c, d) = (ac, bd)$$

for all $a, b, c, d \in \mathbf{K}(n)$. Then ${}^2\mathbf{K}(n)$ may be regarded as a subalgebra of $\mathbf{K}(2n)$ by

$$(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathbf{K}(2n).$$

**Definition** Suppose that $A$ is a finite-dimensional real associative algebra with unit element 1, and that $B, C$ are subalgebras satisfying the following conditions:

1. for any $b \in B, c \in C$, we have $cb = bc$;
2. $A$ is generated as an algebra by $B$ and $C$; and
3. $\dim A = \dim B \dim C$.

Then we say that $A$ is the *(real) tensor product* $B \otimes C$ of $B$ and $C$. For any finite-dimensional real associative algebras $B, C$, there exists an algebra $A$ containing subalgebras $B', C'$ isomorphic to $B, C$ respectively such that $A = B' \otimes C'$; furthermore $A$ is unique up to isomorphism. We may therefore define $B \otimes C$ up to isomorphism for any such $B, C$.

**Lemma 2.3** *If we regard* $\mathbf{C}$ *and* $\mathbf{H}$ *as real algebras, then we have* $\mathbf{C} \otimes \mathbf{H} \cong \mathbf{C}(2)$ *and* $\mathbf{H} \otimes \mathbf{H} \cong \mathbf{R}(4)$.

**Proof** We may identify $\mathbf{C}^2$ with $\mathbf{H}$ as a right complex vector space using the isomorphism $(z, w) \mapsto z + jw$, where $\mathbf{C}$ is identified with $\mathrm{span}\{1, i\} \subset \mathbf{H}$ as before. For any $z \in \mathbf{C}$, $q \in \mathbf{H}$, the maps

$$z_R : \mathbf{H} \to \mathbf{H}; \quad x \mapsto xz \qquad \text{and} \qquad q_L : \mathbf{H} \to \mathbf{H}; \quad x \mapsto qx$$

are right complex linear; the maps

$$\mathbf{C} \to \mathbf{C}(2); \quad z \mapsto z_R \qquad \text{and} \qquad \mathbf{H} \to \mathbf{C}(2); \quad q \mapsto q_L$$

are algebra monomorphisms. Denoting the images of these monomorphisms by $B = \mathbf{C}_R$ and $C = \mathbf{H}_L$ respectively, it is not difficult to show that $\mathbf{C}(2) = B \otimes C$.

For any $q \in \mathbf{H}$, define

$$q_R : \mathbf{H} \to \mathbf{H}; \quad x \mapsto xq$$

and $q_L$ as above. These maps may be considered as (real) linear maps on $\mathbf{R}^4$. The maps

$$\mathbf{H} \to \mathbf{R}(4); \quad q \mapsto q_L \qquad \text{and} \qquad \mathbf{H} \to \mathbf{R}(4); \quad r \mapsto \tilde{r}_R$$

are algebra monomorphisms. Denoting the images of these monomorphisms by $B = \mathbf{H}_L$ and $C = \tilde{\mathbf{H}}_R$ respectively, it is once again easy to show that $\mathbf{R}(4) = B \otimes C$. $\square$

## 2.3 Clifford Algebras and Spin Groups

### 2.3.1 Universal Clifford Algebras

Given a finite-dimensional real vector space $Q$ with nondegenerate bilinear form $q$, we define the *(universal) Clifford algebra* $C(Q, q)$ to be the real associative algebra of largest dimension generated by $Q$ and $\{1\}$ satisfying

$$x^2 = -q(x, x)1 \tag{2.1}$$

for all $x \in Q$, in such a way that $Q$ and $\mathbf{R}$ are embedded isomorphically in $C(Q, q)$ as linear subspaces. In particular, we abbreviate $C(\mathbf{R}^n, \langle \cdot, \cdot \rangle)$ to $C(n)$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^n$. By polarisation of (2.1), we see that if $x, y \in Q$ with $q(x, y) = 0$, then

$$xy = -yx.$$

In particular, if the standard basis of $\mathbf{R}^n$ is $\{e_1, \ldots, e_n\}$ then in $C(n)$ we have the relations

$$e_i^2 = -1, \qquad e_i e_j = -e_j e_i$$

for $1 \leq i, j \leq n$, $i \neq j$. It follows that the dimension of $C(n)$ is $2^n$; a basis is given by

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} : 0 \leq k \leq n, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

We define on $C(Q, q)$ the algebra involution $\widehat{\cdot}$ by

$$\widehat{x} = -x$$

for all $x \in Q$ (and extended by $\widehat{ab} = \widehat{a}\widehat{b}$). In particular, in $C(n)$,

$$(e_{i_1} e_{i_2} \cdots e_{i_k})\widehat{\ } = (-1)^k (e_{i_1} e_{i_2} \cdots e_{i_k})$$

for all $0 \leq k \leq n$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. We define the *even Clifford algebra* $C^0(Q, q)$ by

$$C^0(Q, q) = \{a \in C(Q, q) : \widehat{a} = a\};$$

this is clearly a subalgebra of $C(Q, q)$. A basis for $C^0(n)$ is

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} : 0 \leq k \leq n, k \text{ even}, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

If

$$C^1(Q, q) = \{a \in C(Q, q) : \widehat{a} = -a\},$$

then every element $a \in C(Q, q)$ is uniquely expressible as

$$a = a_0 + a_1$$

with $a_0 \in C^0(Q, q)$ and $a_1 \in C^1(Q, q)$. The elements $a_0$ and $a_1$ are called the *even* and *odd parts of a* respectively. We also define on $C(Q, q)$ the algebra anti-involution $\overline{\cdot}$ by

$$\overline{x} = -x$$

for all $x \in Q$ (and extended by $\overline{ab} = \overline{b}\,\overline{a}$). In particular, in $C(n)$,

$$\overline{e_{i_1} e_{i_2} \cdots e_{i_k}} = (-1)^k (e_{i_k} \cdots e_{i_2} e_{i_1}) = (-1)^{k(k+1)/2} (e_{i_1} e_{i_2} \cdots e_{i_k})$$

for all $0 \leq k \leq n$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. This anti-involution is called *conjugation*.

### 2.3.2 The Clifford Group

For any invertible $g \in C(Q,q)$, define the map $\rho_g : Q \to C(Q,q)$ by

$$\rho_g(x) = gx(\widehat{g})^{-1}$$

for all $x \in Q$, and let

$$\Gamma(Q,q) = \{g \in C(Q,q) : g \text{ invertible}, \rho_g(x) \in Q \text{ for all } x \in Q\}.$$

If $g \in \Gamma(Q,q)$, the map $\rho_g$ is orthogonal, as

$$\begin{aligned} q(\rho_g(x), \rho_g(x)) &= -\left(gx(\widehat{g})^{-1}\right)\left(gx(\widehat{g})^{-1}\right) = \left(gx(\widehat{g})^{-1}\right)^{\widehat{}}\left(gx(\widehat{g})^{-1}\right) \\ &= \widehat{g}\,\widehat{x}g^{-1}gx(\widehat{g})^{-1} = -\widehat{g}x^2(\widehat{g})^{-1} = \widehat{g}q(x,x)(\widehat{g})^{-1} = q(x,x) \end{aligned}$$

for any $x \in Q$. Furthermore, $\rho_g$ is bijective, for if $\rho_g(x) = 0$ then the invertibility of $g$ implies that $x = 0$, whereas the surjectivity of $\rho_g$ follows from the fact that $Q$ is finite-dimensional and from the rank-nullity theorem. In fact $\Gamma(Q,q)$ is a group called the *Clifford group* for $Q$ in $C(Q,q)$. If $a \in Q\backslash\{0\}$, then $a$ is invertible and any $x \in Q$ is expressible as $x = \lambda a + b$ with $\lambda \in \mathbf{R}$ and $q(a,b) = 0$. It follows that

$$\begin{aligned} \rho_a(\lambda a + b) &= a(\lambda a + b)(\widehat{a})^{-1} = -a(\lambda a + b)a^{-1} \\ &= -\lambda a - aba^{-1} = -\lambda a + baa^{-1} = -\lambda a + b, \end{aligned}$$

that is, $a \in \Gamma(Q,q)$ and $\rho_a$ is the reflection in the hyperplane orthogonal to $\mathbf{R}a$. Since every orthogonal map is the product of a finite number of reflections in hyperplanes, we see that $\rho : \Gamma(Q,q) \to O(Q,q); a \mapsto \rho_a$ is a surjective homomorphism and that every element of $\Gamma(Q,q)$ is representable as the product of a finite number of elements of $Q$. In fact $\ker(\rho) = \mathbf{R}^* = \mathbf{R}\backslash\{0\}$.

### 2.3.3 Pin and Spin

For any $a \in C(Q,q)$, define the *norm* of $a$, $N(a)$, by

$$N(a) = \overline{a}\,a.$$

If $g \in \Gamma(Q,q)$, then $g = x_1 \cdots x_k$ for some $x_1, \ldots, x_k \in Q$. It follows that

$$N(g) = \overline{\left(\prod_{i=1}^{k} x_i\right)}\left(\prod_{i=1}^{k} x_i\right) = \left(\prod_{i=k}^{1} \overline{x_i}\right)\left(\prod_{i=1}^{k} x_i\right) = \prod_{i=1}^{k} N(x_i) \in \mathbf{R}^*$$

since $N(x_i) = -x_i^2 = q(x_i, x_i) \in \mathbf{R}^*$ for all $i = 1, \ldots, k$. This implies that the norm $N : \Gamma(Q, q) \to \mathbf{R}^*$ is a homomorphism. We define

$$Pin(Q, q) = \{g \in \Gamma(Q, q) : N(g) = \pm 1\}$$

and

$$Spin(Q, q) = \{g \in \Gamma^0(Q, q) : N(g) = \pm 1\}$$

where $\Gamma^0(Q, q) = \Gamma(Q, q) \cap C^0(Q, q)$ is the "even" subgroup of $\Gamma(Q, q)$. When $Q = \mathbf{R}^n$ and $q$ is the standard inner product, we abbreviate $\Gamma(Q, q)$, $Pin(Q, q)$ and $Spin(Q, q)$ to $\Gamma(n)$, $Pin(n)$ and $Spin(n)$ respectively. It is evident that $Pin(Q, q)$ ($Spin(Q, q)$) is a normal subgroup of $\Gamma(Q, q)$ (respectively $\Gamma^0(Q, q)$) and that

$$\Gamma(Q, q)/Pin(Q, q) \cong \Gamma^0(Q, q)/Spin(Q, q) \cong \mathbf{R}^+.$$

Furthermore, the maps

$$Pin(Q, q) \to O(Q, q); \quad g \mapsto \rho_g \qquad \text{and} \qquad Spin(Q, q) \to SO(Q, q); \quad g \mapsto \rho_g$$

are surjective, the kernel in both cases being isomorphic to $S^0 = \{\pm 1\}$. Note that

$$Spin(Q, q) \subset \{g \in C^0(Q, q) : N(g) = \pm 1\},$$

but the reverse inclusion is not true in general. Topologically, if $n > 1$ then $Spin(n)$ is compact and connected (see [Ps], pp. 226–8) and is thus the connected two-fold covering space of $SO(n)$.

### 2.3.4 Embedding $Spin(n + 1)$ in $C(n)$

Define $\theta : C(n) \to C^0(n + 1)$ by

$$\theta(e_i) = e_i e_{n+1}$$

for $i = 1, \ldots, n$, extended to be an algebra homomorphism. Then $\theta$ is in fact an isomorphism, with

$$\theta^{-1}(e_{i_1} \cdots e_{i_k}) = e_{i_1} \cdots e_{i_k}$$

for all $1 \le i_1 < \cdots < i_k \le n$, $k$ even, and

$$\theta^{-1}(e_{i_1} \cdots e_{i_k} e_{n+1}) = e_{i_1} \cdots e_{i_k}$$

for all $1 \le i_1 < \cdots < i_k \le n$, $k$ odd. Since $Spin(n+1) \subset C^0(n+1)$, we may regard $Spin(n+1)$ as a subgroup of $C(n)$ using the isomorphism $\theta$. Under this identification, we have

$$Spin(n) = \{g \in Spin(n+1) : \widehat{g} = g\}.$$

(This follows from the fact that $\theta$ is norm-preserving on $Q = \mathbf{R}^n$.)

**Lemma 2.4** *If $h \in Spin(n+1)$, then $hx\left(\widehat{h}\right)^{-1} \in Q'$ for all $x \in Q'$, where $Q'$ denotes the subspace $\mathbf{R}1 \oplus \mathbf{R}^n \subset C(n)$.*

**Proof** We may write $h = h_0 + h_1$ where $h_0$ and $h_1$ are the even and odd parts of $h$ respectively. We then have

$$\theta(h) = h_0 + h_1 e_{n+1}; \qquad \theta\left(\widehat{h}\right) = h_0 - h_1 e_{n+1},$$

so

$$\theta(h)e_{n+1} = (h_0 + h_1 e_{n+1})e_{n+1} = e_{n+1}(h_0 - h_1 e_{n+1}) = e_{n+1}\theta\left(\widehat{h}\right).$$

Fix $x \in Q'$; then $x = x_0 + x_1$, where $x_0 \in \mathbf{R}1$ and $x_1 \in \mathbf{R}^n = \operatorname{span}\{e_1, \dots, e_n\}$. Since $\theta(h) \in Spin(n+1) \subset C^0(n+1)$,

$$\theta(h)(-x_0 e_{n+1} + x_1) = x'\theta(h)$$

for some $x' \in \mathbf{R}^{n+1} \subset C(n+1)$, since $-x_0 e_{n+1} + x_1 \in \mathbf{R}^{n+1} \subset C(n+1)$. Consequently

$$\begin{aligned}
\theta(h)\theta(x) &= \theta(h)(x_0 + x_1 e_{n+1}) \\
&= \theta(h)(-x_0 e_{n+1} + x_1)e_{n+1} \\
&= x'\theta(h)e_{n+1} \\
&= x' e_{n+1}\theta\left(\widehat{h}\right) \\
&= \theta(x'')\theta\left(\widehat{h}\right)
\end{aligned}$$

for some $x'' \in Q'$. It follows that $hx\left(\widehat{h}\right)^{-1} \in Q'$ for all $x \in Q'$, as claimed. Note that, since $Spin(n+1)$ preserves norms, the map

$$Q' \to Q'; \quad x \mapsto hx\left(\widehat{h}\right)^{-1}$$

is a rotation of $Q'$. □

**Lemma 2.5** *Any element of $Spin(n+1) \subset C(n)$ is expressible in the form $zg$ for some $z \in S^n = \{x \in Q' : \overline{x}x = 1\}$ and some $g \in Spin(n)$.*

**Proof** If $h \in Spin(n+1)$, then $h\left(\widehat{h}\right)^{-1} \in S^n$ since $1 \in S^n$. Let $z \in Q'$ satisfy the condition $z^2 = h\left(\widehat{h}\right)^{-1}$ (noting that $(Q')^2 = Q'$) and let $g = \widehat{z}h$. Clearly $z \in S^n$, so $\widehat{z} = \overline{z} = z^{-1}$ and

$$g(\widehat{g})^{-1} = z^{-1}h\left(\widehat{h}\right)^{-1}z^{-1} = z^{-1}z^2z^{-1} = 1,$$

that is, $g \in Spin(n)$ and $h = zg$. $\square$

Consider the sequence

$$Spin(n) \xrightarrow{\iota} Spin(n+1) \xrightarrow{\mu} S^n,$$

where $\iota$ is the inclusion map and $\mu$ is the map $Spin(n+1) \to S^n$; $h \mapsto h\left(\widehat{h}\right)^{-1}$. We claim that the fibres of $\mu$ are the left cosets of $Spin(n)$ in $Spin(n+1)$. Indeed, if $x_1, x_2 \in Q'$ satisfy $x_1^2 = x_2^2$, then it is easy to see that $x_1 = \pm x_2$. If $h_1, h_2 \in Spin(n+1)$ satisfy $\mu(h_1) = \mu(h_2) = z^2$ for some $z \in S^n$, then writing $h_1 = z_1g_1$ and $h_2 = z_2g_2$ for $z_1, z_2 \in S^n$ and $g_1, g_2 \in Spin(n)$, we see that $z_1 = \pm z$, $z_2 = \pm z$ and $h_1^{-1}h_2 \in Spin(n)$. Since $\mu$ is surjective, we may view $Spin(n)$ as the stabiliser subgroup of any given element of $S^n$ under the transitive action of $Spin(n+1)$, that is, $h(x) = hx\left(\widehat{h}\right)^{-1}$.

### 2.3.5 Construction of Low-Dimensional Clifford Algebras and Spin Groups

**Lemma 2.6** *The Clifford algebras $C(n), n = 0, 1, 2, 3, 4$ are isomorphic to* $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, ${}^2\mathbf{H}$, $\mathbf{H}(2)$ *respectively.*

**Proof** $C(0) \cong \mathbf{R}$ is obvious. For the other cases,

1. In $\mathbf{C}$, set $e_1 = i$.

2. In $\mathbf{H}$, set $e_1 = i$, $e_2 = k$.

3. In ${}^2\mathbf{H}$, set $e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $e_2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$, $e_3 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$.

4. In $\mathbf{H}(2)$, set $e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $e_2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$, $e_3 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

It is easy to verify that the result holds with these identifications. □

**Lemma 2.7** *For all $n \geq 4$, $C(n) \cong C(n-4) \otimes C(4) \cong C(n-4) \otimes \mathbf{H}(2)$.*

**Proof** Consider $C = C(4)$ as a subalgebra of $C(n)$, generated by the elements 1, $e_1$, $e_2$, $e_3$, $e_4$ of $C(n)$. Let $a = e_1e_2e_3e_4$ and let $B$ denote the subalgebra of $C(n)$ generated by the elements $1, ae_5, \ldots, ae_n$ of $C(n)$. It is evident that $B$ is isomorphic to $C(n-4)$, since $(ae_i)(ae_j) = e_ie_j$ for any $5 \leq i, j \leq n$. Furthermore, every element of $B$ commutes with every element of $C$, since $e_i$ anticommutes with $a$ for $1 \leq i \leq 4$ and commutes with $a$ for $5 \leq i \leq n$. Clearly $C(n)$ is generated by $B$ and $C$, and

$$\dim C(n) = 2^n = 2^{n-4}2^4 = \dim(C(n-4)) \dim(C(4)).$$

The result follows immediately. □

Lemmas 2.3 and 2.7 enable us to identify the following low-dimensional Clifford algebras.

**Corollary**

$$\begin{array}{lclcll} C(5) & \cong & \mathbf{C} \otimes \mathbf{H}(2) & \cong & \mathbf{C}(4); & \\ C(6) & \cong & \mathbf{H} \otimes \mathbf{H}(2) & \cong & \mathbf{R}(8); & \\ C(7) & \cong & {}^2\mathbf{H} \otimes \mathbf{H}(2) & \cong & {}^2\mathbf{R}(8); & \quad \textit{and} \\ C(8) & \cong & \mathbf{H}(2) \otimes \mathbf{H}(2) & \cong & \mathbf{R}(16). & \end{array}$$

By induction, the spaces $C(n)$ are isomorphic to $\mathbf{K}(2^m)$ for some $m$ and $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, ${}^2\mathbf{R}$ or ${}^2\mathbf{H}$ depending on $n$. The corresponding space $\mathbf{K}^{2^m}$ is called the *spinor space* of $C(n)$. It is not difficult to see that conjugation on $C(n) \cong \mathbf{K}(2^m)$ is in fact the adjoint operation on $\mathbf{K}(2^m)$. This is a potential source of confusion, for $\overline{e_i}$ (conjugation in $C(n)$) corresponds to $(\overline{e_i})^t$ (conjugation in $\mathbf{K}(2^m)$). (In [Ps], Porteous uses $x^-$ for the Clifford algebra conjugation in order to avoid this ambiguity.)

**Lemma 2.8** *For $n \leq 5$, $Spin(n) \cong \{g \in C^0(n) : N(g) = \pm 1\}$.*

The proof of this result may be found in [Ps], p. 147.

**Corollary**

$$\begin{array}{ll} \mathit{Spin}(1) \cong O(1) \cong S^0, & \mathit{Spin}(2) \cong U(1) \cong S^1, \\ \mathit{Spin}(3) \cong \mathit{Sp}(1) \cong S^3, & \mathit{Spin}(4) \cong \mathit{Sp}(1) \times \mathit{Sp}(1) \cong S^3 \times S^3, \\ \mathit{Spin}(5) \cong \mathit{Sp}(2), & \mathit{Spin}(6) \subset U(4). \end{array}$$

(We recall that $\mathit{Sp}(n)$ denotes the group of isometries of $\mathbf{H}^n$ with respect to the standard norm $|(x_1, \dots, x_n)| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$.)

We now give an explicit description of $\mathit{Spin}(2)$ and $\mathit{Spin}(4)$. In the case of $\mathit{Spin}(2)$, we have $Q' = \mathbf{R}^2$ which is identified with $\mathbf{C}$. The unit complex number $g \in U(1) \cong \mathit{Spin}(2)$ acts as a rotation of $Q'$ by

$$y \mapsto gy(\widehat{g})^{-1} = gyg = g^2 y.$$

In the case of $\mathit{Spin}(4)$, we have $Q' = \mathbf{R}^4$, identified with $\left\{ \begin{pmatrix} y & 0 \\ 0 & \tilde{y} \end{pmatrix} : y \in \mathbf{H} \right\}$. Under this identification, if $\begin{pmatrix} q & 0 \\ 0 & r \end{pmatrix} \in {}^2\mathbf{H}$ then

$$\widehat{\begin{pmatrix} q & 0 \\ 0 & r \end{pmatrix}} = \begin{pmatrix} \widehat{r} & 0 \\ 0 & \widehat{q} \end{pmatrix}.$$

The element $\begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix} \in \mathit{Spin}(4)$, where $q, r \in S^3$, acts orthogonally on $\begin{pmatrix} y & 0 \\ 0 & \tilde{y} \end{pmatrix}$ by

$$\begin{pmatrix} y & 0 \\ 0 & \tilde{y} \end{pmatrix} \mapsto \begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & \tilde{y} \end{pmatrix} \widehat{\begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix}}^{-1} = \begin{pmatrix} qy\overline{r} & 0 \\ 0 & \widehat{r}\tilde{y}\tilde{q} \end{pmatrix},$$

that is, $y \mapsto qy\overline{r}$.

## 2.4 Octonions and Triality

### 2.4.1 Fundamental Properties of O

The *octonion algebra* $\mathbf{O}$ is usually defined to be $\mathbf{H}^2$ equipped with the product

$$(x, y)(x', y') = (xx' - \overline{y'}y, y\overline{x'} + y'x)$$

and conjugation

$$\overline{(x,y)} = (\overline{x}, -y)$$

for all $x, y, x', y' \in \mathbf{H}$, however we do not follow this route but instead present an equivalent formulation which does not require the use of quaternions and more importantly clarifies the link between octonions, Clifford algebras and $Spin(8)$.

Consider the Lie algebra $\mathfrak{so}(8)$ of $Spin(8)$ (and $SO(8)$) spanned by

$$\{X_{jk} : 0 \leq j < k \leq 7\}, \quad \text{where } X_{jk} = E_{jk} - E_{kj} \in \mathbf{R}(8).$$

Here $E_{jk}$ is the elementary $8 \times 8$ matrix with 1 in the $(j,k)$-position and 0 elsewhere. A maximal torus $\mathfrak{t}$ is spanned by $\{Y_j\}_{j=0}^{3}$ where

$$Y_j = X_{2j,2j+1}.$$

The (complex infinitesimal) roots are

$$\{i(\pm Y_j^* \pm Y_k^*)\}_{1 \leq k < j \leq 4}$$

where $Y_4 = Y_0$. An ordered basis is given by

$$\{i(Y_2^* - Y_3^*), i(Y_3^* + Y_4^*), i(Y_3^* - Y_4^*), i(Y_1^* - Y_2^*)\};$$

we denote these roots by $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ respectively. The associated Dynkin diagram is depicted in Figure 2.1.

In Appendix A we describe the construction of a map $\Theta$, known as the *triality automorphism of* $\mathfrak{so}(8)$. This map is a Lie algebra automorphism of order 3 whose

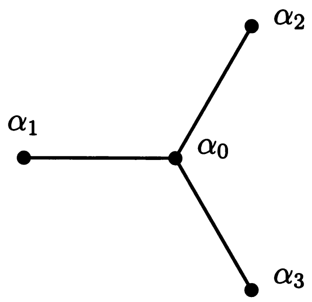


Figure 2.1: Dynkin diagram for $\mathfrak{so}(8)$

dual map preserves $\alpha_0$ and cyclically permutes $\alpha_1, \alpha_2, \alpha_3$. Let $\{e_j\}_{j=0}^7$ denote the standard basis for $\mathbf{R}^8$. Define a map $\nu : \mathbf{R}^8 \to \mathbf{R}^8$ by setting

$$\nu(e_0) = I \qquad \text{and} \qquad \nu(e_k) = 2\Theta(X_{0,k}), \quad k = 1, \dots, 7$$

and extending to make $\nu$ linear. Based on the calculations in Appendix A, we find that

$$\nu(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & -x_7 \\ x_1 & x_0 & -x_3 & x_2 & x_5 & -x_4 & x_7 & -x_6 \\ x_2 & x_3 & x_0 & -x_1 & x_6 & -x_7 & -x_4 & x_5 \\ x_3 & -x_2 & x_1 & x_0 & x_7 & x_6 & -x_5 & -x_4 \\ x_4 & -x_5 & -x_6 & -x_7 & x_0 & x_1 & x_2 & x_3 \\ x_5 & x_4 & x_7 & -x_6 & -x_1 & x_0 & x_3 & -x_2 \\ x_6 & -x_7 & x_4 & x_5 & -x_2 & -x_3 & x_0 & x_1 \\ x_7 & x_6 & -x_5 & x_4 & -x_3 & x_2 & -x_1 & x_0 \end{pmatrix}$$

for all $(x_0, \dots, x_7) \in \mathbf{R}^8$. Furthermore

$$\Theta(X_{j,k}) = \frac{1}{2}\nu(e_k)\nu(e_j) \tag{2.2}$$

for all $0 \le j < k \le 7$. Let $\mathbf{Y} = \nu(\mathbf{R}^8)$. We shall investigate further properties of $\Theta$ in Section 2.4.4. (As an aside, we note that $\mathbf{R}(8)$ may be regarded as a non-universal Clifford algebra with $Q = \mathbf{R}^7$. The term *non-universal* means that the Clifford algebra has dimension less than the maximal $2^7 = 128$. Conjugation in the Clifford algebra corresponds to transposition in $\mathbf{R}(8)$ and the subspace $Q'$ corresponds to $\mathbf{Y}$. For a more detailed discussion, see Chapter 19 of [Ps].)

We now construct the octonion algebra. For $x, y \in \mathbf{R}^8$, define the (bilinear) product $xy \in \mathbf{R}^8$ by

$$xy = \nu(x)y.$$

If $e = e_0 = (1, 0, 0, 0, 0, 0, 0, 0) \in \mathbf{R}^8$, we have

$$ex = \nu(e)x = Ix = x = \nu(x)e = xe,$$

so $e$ is the identity element of the algebra $\mathbf{R}^8$. It is easy to verify that

$$(\nu(x))^t\nu(x) = |x|^2 I \tag{2.3}$$

for all $x \in \mathbf{R}^8$. If $x, y \in \mathbf{R}^8$ we have

$$|xy|^2 = |\nu(x)y|^2 = y^t(\nu(x))^t\nu(x)y = y^t |x|^2 Iy = |x|^2 |y|^2,$$

implying that $\mathbf{R}^8$ is a normed division algebra. We henceforth refer to $\mathbf{R}^8$ with the above product as the *octonion* or *Cayley algebra* and denote it by $\mathbf{O}$. We also define, for $x \in \mathbf{O}$, the element $\overline{x} \in \mathbf{O}$ given by

$$\overline{x} = (\nu(x))^t e.$$

Explicitly, if $x = (x_0, x_1, \dots, x_7)$, then $\overline{x} = (x_0, -x_1, \dots, -x_7)$. We call $x_0$ the *real part* of $x$ and $(0, x_1, \dots, x_7)$ the *imaginary part* of $x$, denoted by $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ respectively. Note that $\overline{e} = e$, $\overline{\overline{x}} = x$, $\mathrm{Re}(x)e = (x + \overline{x})/2$, $\mathrm{Im}(x) = (x - \overline{x})/2$ for all $x \in \mathbf{O}$ and that $\overline{x} = -x$ if and only if $x = \mathrm{Im}(x)$. We claim that

$$\langle x, y \rangle e = \frac{1}{2}(\overline{x}y + \overline{y}x)$$

for all $x, y \in \mathbf{O}$; this follows from

$$\overline{x}y + \overline{y}x = \overline{x}(ye) + \overline{y}(xe) = (\nu(x))^t\nu(y)e + (\nu(y))^t\nu(x)e = 2\langle x, y \rangle e \tag{2.4}$$

by polarisation of (2.3). Since $(\nu(x))^t\nu(x) = |x|^2 I = \nu(x)(\nu(x))^t$, we have

$$|x|^2 e = \overline{x}x = x\overline{x}.$$

It follows that

$$x^{-1} = |x|^{-2}\,\overline{x}$$

for all $x \in \mathbf{O}$. Furthermore, if $x$ and $y$ are imaginary then they anticommute if and only if they are orthogonal.

**Definition** For $x, y, z \in \mathbf{O}$, we define the *scalar triple product* of $x$, $y$ and $z$ to be

$$\{x, y, z\} = \langle \overline{x}, yz \rangle.$$

**Lemma 2.9** *The quantity $\{x, y, z\}$ is invariant under even permutations of the entries, that is,*

$$\{x, y, z\} = \{y, z, x\} = \{z, x, y\}$$

*for all $x, y, z \in \mathbf{O}$.*

**Proof** We have

$$\{x, y, z\} = \langle \overline{x}, yz \rangle = \overline{x}^t \nu(y) z = \overline{x}^t (\nu(\overline{y}))^t z = (\nu(\overline{y})\overline{x})^t z = \langle \overline{y}\,\overline{x}, z \rangle = \{\overline{z}, \overline{y}, \overline{x}\}.$$

If $x = x_0 e + x_1$ where $x_0 = \mathrm{Re}(x)$ and $x_1 = \mathrm{Im}(x)$, then

$$\{y, z, x_0 e\} = \langle \overline{y}, z x_0 e \rangle = x_0 \langle \overline{y}, z \rangle = \langle z, \overline{y}\,\overline{x_0 e} \rangle = \{\overline{z}, \overline{y}, \overline{x_0 e}\}$$

and

$$\begin{aligned} \{y, z, x_1\} - \{\overline{z}, \overline{y}, \overline{x_1}\} &= \{\overline{x_1}, \overline{z}, \overline{y}\} - \{x_1, y, z\} \\ &= \{\overline{x_1}, \overline{z}, \overline{y}\} + \{\overline{x_1}, y, z\} \\ &= \langle x_1, \overline{z}\,\overline{y} + yz \rangle \\ &= \langle x_1, 2 \langle \overline{z}, y \rangle e \rangle \\ &= 0. \end{aligned}$$

By linearity it follows that

$$\{y, z, x\} = \{\overline{z}, \overline{y}, \overline{x}\} = \{x, y, z\},$$

and by repeating the same argument we also have $\{x, y, z\} = \{z, x, y\}$. □

There appears to be nothing of interest to say about odd permutations of the entries of the scalar triple product.

**Corollary** *For all $x, y \in \mathbf{O}$,*

$$\overline{xy} = \overline{y}\,\overline{x}.$$

**Proof** Set $z = e$. By Lemma 2.9, we have $\{e, x, y\} = \{x, y, e\}$. By (2.4),

$$e(xy) + (\overline{xy})\overline{e} = x(ye) + (\overline{ye})\overline{x},$$

that is, $xy + \overline{xy} = xy + \overline{y}\,\overline{x}$. The corollary follows. □

We also have

$$\overline{x}(xy) = \nu(\overline{x})\nu(x)y = (\nu(x))^t \nu(x) y = |x|^2 y = (\overline{x}x)y$$

for all $x, y \in \mathbf{O}$, implying that

$$\begin{aligned} x(xy) &= (x + \overline{x})xy - \overline{x}(xy) \\ &= ((x + \overline{x})x)y - (\overline{x}x)y \qquad \text{since } x + \overline{x} \in \mathbf{R}e \\ &= x^2 y, \end{aligned}$$

that is, $\mathbf{O}$ is an *alternative* algebra.

Pick $i, j \in \mathrm{Im}(\mathbf{O})$ such that $|i| = |j| = 1$ and $\langle i, j \rangle = 0$. Then if $k = ij$, we have $k = \mathrm{Im}(k)$, for we must have $ij + ji = 0$, implying that

$$\overline{k} = (ij)^{-} = \overline{j}\,\overline{\imath} = ji = -ij = -k.$$

Furthermore,

$$\langle i, k \rangle e = \frac{1}{2}\left(\overline{\imath}(ij) + (ij)^{-} i\right) = \frac{1}{2}|i|^2 (j + \overline{j}) = 0,$$

whence $\langle i, k \rangle = \langle j, k \rangle = 0$ by a similar argument. Since $|k| = |i|\,|j| = 1$, we see that the elements $\{i, j\}$ generate a subalgebra of $\mathbf{O}$ isomorphic to $\mathbf{H}$ which is therefore associative. Nevertheless, the algebra $\mathbf{O}$ is not associative, for if $i, j$ are as above and $l \in \mathrm{Im}(\mathbf{O})\backslash\{0\}$ is chosen such that $l$ is orthogonal to $i, j$ and $ij$, then

$$ij + ji = il + li = jl + lj = (ij)l + l(ij) = 0,$$

whence

$$i(jl) + (ij)l = -i(lj) - l(ij) = (i^2 + l^2)j - (i + l)(ij + lj) \tag{2.5}$$
$$= (i + l)^2 j - (i + l)((i + l)j) = 0 \tag{2.6}$$

by the alternativity of $\mathbf{O}$. If $\mathbf{O}$ were in fact associative, (2.5) would imply that $(ij)l = 0$ which contradicts the fact that $\mathbf{O}$ is a division algebra.

We have noted that $\mathrm{Re}(x)e = (x + \overline{x})/2$ for all $x \in \mathbf{O}$, thus by (2.4),

$$\mathrm{Re}(\overline{x}y) = \frac{1}{2}\left(\overline{x}y + (\overline{x}y)^{-}\right) = \frac{1}{2}(\overline{x}y + \overline{y}x) = \langle x, y \rangle e.$$

A simple calculation shows that $\langle \overline{x}, y \rangle = \langle x, \overline{y} \rangle$ for all $x, y \in \mathbf{O}$, so

$$\mathrm{Re}(xy) = \langle \overline{x}, y \rangle = \langle x, \overline{y} \rangle = \langle \overline{y}, x \rangle = \mathrm{Re}(yx)$$

for all $x, y \in \mathbf{O}$. Also

$$\mathrm{Re}(x(yz)) = \langle \overline{x}, yz \rangle = \{x, y, z\} = \{z, x, y\} = \langle \overline{z}, xy \rangle = \mathrm{Re}(z(xy)) = \mathrm{Re}((xy)z)$$

for all $x, y, z \in \mathbf{O}$. Another useful fact is that if $x \in \mathbf{O}$ satisfies $xy = yx$ for all $y \in \mathbf{O}$ then $x \in \mathbf{R}e$. To see this, choose a subalgebra $A$ of $\mathbf{O}$ containing $x$ isomorphic to $\mathbf{H}$ and use the analogous (easily verified) result for $\mathbf{H}$.

**Lemma 2.10**

*(i) For all $a, b \in \mathbf{O}$, $(ab)\overline{a} = a(b\overline{a}) = ab\overline{a}$. If $b$ is imaginary, then so is $ab\overline{a}$. If $a$ is also imaginary and $|a| = 1$, the map $\mathbf{R}^7 \to \mathbf{R}^7 : b \mapsto -ab\overline{a}$ is the reflection in the hyperplane $\{\mathbf{R}a\}^\perp$.*

*(ii) For all $a, b \in \mathbf{O}$, $(ab)a = a(ba) = aba$.*

*(iii) (Moufang identity) For all $a, b, c \in \mathbf{O}$, $a(bc)a = (ab)(ca)$.*

*(iv) If $r \in \mathbf{O}$ satisfies $r(xy) = (rx)y$ for all $x, y \in \mathbf{O}$, then $r \in \mathbf{R}e$.*

**Proof** For any $a, b, c \in \mathbf{O}$, define their *associator* by

$$[a, b, c] = (ab)c - a(bc).$$

The alternativity property of $\mathbf{O}$ may then be expressed as $[a, a, b] = 0$ for all $a, b \in \mathbf{O}$. We have also shown that $[\overline{a}, a, b] = 0$ for all $a, b \in \mathbf{O}$. For any $a, b, c \in \mathbf{O}$, the trilinearity of $[\,\cdot\,,\cdot\,,\cdot\,]$ implies that

$$0 = [a + c, a + c, b] = [a, a, b] + [c, c, b] + [a, c, b] + [c, a, b],$$

so $[a, c, b] = -[c, a, b]$. Since

$$\overline{[a, b, c]} = -\left[\overline{c}, \overline{b}, \overline{a}\right],$$

we have $[b, c, a] = -[b, a, c]$ (after replacing $a, b, c$ with their conjugates). As a result, the associator is invariant under even permutations of its arguments and changes sign under odd such permutations. It follows immediately that $[a, b, \overline{a}] = 0$ and $[a, b, a] = 0$, establishing (ii) and the first part of (i). To prove the rest of (i), suppose that $b$ is imaginary; then

$$\mathrm{Re}(ab\overline{a}) = \mathrm{Re}((ab)\overline{a}) = \mathrm{Re}(\overline{a}(ab)) = \mathrm{Re}\left(|a|^2 b\right) = |a|^2 \,\mathrm{Re}(b) = 0,$$

so $ab\overline{a}$ is imaginary. If $a$ is also imaginary and $|a| = 1$, then defining $\rho_a : \mathbf{R}^7 \to \mathbf{R}^7$ to be the linear map $b \mapsto -ab\overline{a}$ we have

$$\rho_a(a) = -aa\overline{a} = -a$$

and

$$\rho_a(b) = -ab\overline{a} = ba\overline{a} = b$$

for all $b \in \mathbf{R}^7$ orthogonal to $a$. This completes the proof of (i).

Now $c(aba) = ((ca)b)a$ for all $a, b, c \in \mathbf{O}$, for

$$\begin{aligned}
c(aba) - ((ca)b)a &= c(a(ba)) - ((ca)b)a \\
&= -[c, a, ba] + (ca)(ba) - (ca)(ba) - [ca, b, a] \\
&= [c, ba, a] + [b, ca, a] \\
&= (c(ba))a - c\left(ba^2\right) + (b(ca))a - b\left(ca^2\right) \\
&= \left[c, b, a^2\right] - (cb)a^2 + \left[b, c, a^2\right] - (bc)a^2 + (b(ca) + c(ba))a \\
&= (b(ca) + c(ba) - (cb)a - (bc)a)a \\
&= -\left([c, b, a] + [b, c, a]\right) a \\
&= 0.
\end{aligned}$$

Then

$$\begin{aligned}
a(bc)a - (ab)(ca) &= ((ab)c)a - [a, b, c]a - ((ab)c)a + [ab, c, a] \\
&= -[a, b, c]a - [c, ab, a] \\
&= -[a, b, c]a - (c(ab))a + c(aba) \\
&= -[a, b, c]a - (c(ab))a + ((ca)b)a, \quad \text{by the above claim} \\
&= \left([c, a, b] - [a, b, c]\right) a \\
&= 0
\end{aligned}$$

proving (iii).

In order to establish (iv), we first show that the image of the associator contains $\mathrm{Im}(\mathbf{O})$, that is, $\mathrm{Im}(\mathbf{O}) \subseteq [\mathbf{O}, \mathbf{O}, \mathbf{O}]$. Suppose that we are given $n \in \mathrm{Im}(\mathbf{O})$; write $n = \alpha m$ with $\alpha \in \mathbf{R}$ and $|m| = 1$. Choose $l \in \mathrm{Im}(\mathbf{O})$ with $l \perp m$ and $|l| = 1$ and set $k = lm$. Pick $i \in \mathrm{Im}(\mathbf{O})$ with $i \perp l$, $i \perp m$, $i \perp k$ and $|i| = 1$ and set $j = ki$. It is easy to show that $j, k \in \mathrm{Im}(\mathbf{O})$, $|k| = |j| = 1$ and that $k \perp l$, $k \perp m$, $i \perp j$ and $k \perp j$. Also, by the Moufang identity (iii), we have

$$\langle j, l\rangle e = -jl - lj = (ki)(mk) + (km)(ik) = k(im + mi)k = 0,$$

that is, $j \perp l$. By (2.5), $(ij)l + i(jl) = 0$, so $[i, j, l] = 2(ij)l = 2kl = 2m$. Finally, $[\frac{\alpha}{2}i, j, l] = \alpha m = n$, so that $\mathrm{Im}(\mathbf{O}) \subseteq [\mathbf{O}, \mathbf{O}, \mathbf{O}]$ as claimed. (In fact the reverse inclusion holds as well, for we have seen that $\mathrm{Re}([a, b, c]) = 0$ for all $a, b, c \in \mathbf{O}$.)

Now we note that polarisation of the Moufang identity (iii) gives

$$(ca)(bd) + (da)(bc) = (c(ab))d + (d(ab))c. \tag{2.7}$$

We now prove that

$$d[a,b,c]-[a,b,c]d+[ab,c,d]+[bc,a,d]+[ca,b,d]=0 \tag{2.8}$$

for all $a,b,c,d \in \mathbf{O}$. In fact

$$\begin{aligned}
d[a,b,c] &= d((ab)c)-d(a(bc))\\
&= -[d,ab,c]+(d(ab))c+[d,a,bc]-(da)(bc)\\
&= -[ab,c,d]-[bc,a,d]-(c(ab))d+(ca)(bd) \quad \text{by (2.7)}\\
&= -[ab,c,d]-[bc,a,d]+[c,a,b]d-((ca)b)d+(ca)(bd)\\
&= -[ab,c,d]-[bc,a,d]+[a,b,c]d-[ca,b,d].
\end{aligned}$$

Now suppose that $[r,x,y]=0$ for all $x,y \in \mathbf{O}$. Setting $d=r$ in (2.8) we have

$$r[a,b,c]-[a,b,c]r=0$$

for all $a,b,c \in \mathbf{O}$. Since $\mathrm{Im}(\mathbf{O}) \subseteq [\mathbf{O},\mathbf{O},\mathbf{O}]$, it follows that $r$ commutes with all imaginary octonions, hence all octonions. By the remarks preceding the statement of this lemma, $r \in \mathbf{R}e$. $\square$

### 2.4.2 *Spin*(8)

In the corollary to Lemma 2.7 we saw that the Clifford algebra $C(7)$ is isomorphic to ${}^2\mathbf{R}(8)$. As in Section 2.3.4 we may consider $Spin(8)$ as being embedded in $C(7)$. In this section we realise $Spin(8)$ as a subgroup of ${}^2\mathbf{R}(8)$. In fact, by the orthogonality of the Clifford groups (see Section 2.3.2) and the fact that $Spin(8)$ is connected, we may realise $Spin(8)$ as a subgroup of $SO(8)\times SO(8)$.

Let $\{e,e_1,\dots,e_7\}$ denote the standard basis of $\mathbf{R}^8$. It is easy to verify that $\{\nu(e_i)\}_{i=1}^7$ is a set of pairwise anticommuting matrices of determinant 1. (This is consistent with the interpretation of $\mathbf{R}(8)$ as a nonuniversal Clifford algebra described in Section 2.4.1; it is also consistent with equation (2.2).) When $\mathbf{Y}$ is embedded in ${}^2\mathbf{R}(8) \cong C(7)$ by

$$y \mapsto \begin{pmatrix} y & 0\\ 0 & y^t \end{pmatrix}$$

for $y \in \mathbf{Y}$, it is clear that the subspace $Q' = \mathbf{R} \oplus Q$ of $C(7)$ may be identified with

$\mathbf{Y} \subset {}^2\mathbf{R}(8)$. The element $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$ of ${}^2\mathbf{R}(8)$ satisfies

$$\widehat{\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}} = \begin{pmatrix} h & 0 \\ 0 & g \end{pmatrix},$$

hence the element $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \in Spin(8)$ acts orthogonally on $\mathbf{Y}$ by

$$\begin{pmatrix} y & 0 \\ 0 & y^t \end{pmatrix} \mapsto \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^t \end{pmatrix} \widehat{\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}}^{-1} = \begin{pmatrix} gyh^t & 0 \\ 0 & hy^tg^t \end{pmatrix},$$

since $g$ and $h$ are necessarily in $SO(8)$. That is,

$$Spin(8) = \{(g, h) : g, h \in SO(8), gyh^t \in \mathbf{Y} \text{ for all } y \in \mathbf{Y}\},$$

however for technical reasons we now make a slightly different characterisation of $Spin(8)$. Define the *companion involution* $\mathbf{R}(8) \to \mathbf{R}(8)$; $g \mapsto \breve{g}$ by

$$\breve{g} = JgJ$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -I_7 \end{pmatrix} \in \mathbf{R}(8);$$

we call $\breve{g}$ the *companion* of $g$. It is obvious that

$$\breve{g}x = \overline{g\overline{x}}$$

for all $g \in \mathbf{R}(8)$, $x \in \mathbf{O}$. This implies that if $g \in SO(8)$, then $\breve{g} \in SO(8)$. It also implies that $g = \breve{g}$ if and only if $ge = e$, in which case

$$g = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$$

for some $h \in SO(7)$. We then write

$$Spin(8) = \left\{ \begin{pmatrix} g_0 & 0 \\ 0 & \breve{g}_1 \end{pmatrix} : g_0, g_1 \in SO(8), g_0 y \breve{g}_1^{-1} \in \mathbf{Y} \text{ for all } y \in \mathbf{Y} \right\} \subset \mathbf{R}(16).$$

The homomorphism $Spin(8) \to SO(8)$; $\begin{pmatrix} g_0 & 0 \\ 0 & \breve{g}_1 \end{pmatrix} \mapsto g_0$ is one of the possible projections of $Spin(8)$ onto $SO(8)$, thus has kernel

$$\left\{ \begin{pmatrix} I_8 & 0 \\ 0 & I_8 \end{pmatrix}, \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix} \right\}.$$

### 2.4.3 Triality

The orthogonal action of $Spin(8)$ on $S^7$ is given by

$$\begin{pmatrix} g_0 & 0 \\ 0 & \check{g}_1 \end{pmatrix} x = g_0 \nu(x) \check{g}_1^{-1} e$$

for all $x \in S^7 \cong \{x \in \mathbf{O} : |x| = 1\}$. Consequently, for $\begin{pmatrix} g_0 & 0 \\ 0 & \check{g}_1 \end{pmatrix} \in Spin(8)$, there exists a unique $g_2 \in SO(8)$ such that

$$g_0 y \check{g}_1^{-1} e = \check{g}_2 y e$$

for all $y \in \mathbf{Y}$. We call the ordered triple $(g_0, g_1, g_2)$ a *$\theta$-triad* of $SO(8)$, and define $\theta : Spin(8) \to Spin(8)$ by

$$\theta \begin{pmatrix} g_0 & 0 \\ 0 & \check{g}_1 \end{pmatrix} = \begin{pmatrix} g_1 & 0 \\ 0 & \check{g}_2 \end{pmatrix}.$$

Then as Lemma 2.11 below implies, $\theta$ is a well-defined automorphism of order 3. It is known as the *triality* automorphism of $Spin(8)$. In fact $\theta$ is closely related to the map $\Theta$ defined in Section 2.4.1; we shall investigate this connection in Section 2.4.4 below.

**Lemma 2.11** *Let $(g_0, g_1, g_2)$ be a $\theta$-triad of $SO(8)$. Then $(g_1, g_2, g_0)$, $(g_2, g_0, g_1)$ and $(g_0^{-1}, g_1^{-1}, g_2^{-1})$, $(g_1^{-1}, g_2^{-1}, g_0^{-1})$, $(g_2^{-1}, g_0^{-1}, g_1^{-1})$ are $\theta$-triads of $SO(8)$. Furthermore, $(\check{g}_1, \check{g}_0, \check{g}_2)$ is also a $\theta$-triad of $SO(8)$.*

**Proof** If $y, z \in \mathbf{Y}$ then

$$g_0 y \check{g}_1^{-1} z e = \left(g_0 y \check{g}_1^{-1} e\right)(ze) = (\check{g}_2 y e)(ze)$$

since $g_0 y \check{g}_1^{-1} \in \mathbf{Y}$. Now if $x \in \mathbf{Y}$, then

$$\begin{aligned} \langle \overline{ye}, (\check{g}_1^{-1} ze)(xe) \rangle &= \left\{ye, \check{g}_1^{-1} ze, xe\right\} = \left\{xe, ye, \check{g}_1^{-1} ze\right\} = \langle \overline{xe}, y \check{g}_1^{-1} ze \rangle \\ &= \left\langle g_0 \overline{xe}, g_0 y \check{g}_1^{-1} ze \right\rangle = \langle g_0 \overline{xe}, (\check{g}_2 ye)(ze) \rangle = \left\langle \overline{\check{g}_0 xe}, \overline{g_2 \overline{ye}}(ze) \right\rangle \\ &= \left\{\check{g}_0 xe, \overline{g_2 \overline{ye}}, ze\right\} = \left\{\overline{g_2 \overline{ye}}, ze, \check{g}_0 xe\right\} = \langle g_2 \overline{ye}, z \check{g}_0 xe \rangle \\ &= \left\langle \overline{ye}, g_2^{-1} z \check{g}_0 xe \right\rangle \end{aligned}$$

by the orthogonality of $g_0$ and $g_2$. Since this is true for all $y \in \mathbf{Y}$ we have

$$\left(\check{g}_1^{-1} ze\right)(xe) = g_2^{-1} z \check{g}_0 xe$$

for all $z, x \in \mathbf{Y}$. It follows that $g_2^{-1} z \check{g}_0 \in \mathbf{Y}$ for all $z \in \mathbf{Y}$ and, by setting $x = I$, that $(g_2^{-1}, g_0^{-1}, g_1^{-1})$ is a $\theta$-triad of $SO(8)$. (Note that $(\check{g})^{-1} = (g^{-1})\check{}$ for all $g \in SO(8)$.) Repeating this argument (four times) completes the proof of the first claim. Now $\mathbf{Y}$ is closed under transposition, so

$$(g_0 y \check{g}_1^{-1})^t = \check{g}_1 y^t g_0^{-1} \in \mathbf{Y}$$

for all $y \in \mathbf{Y}$; it follows that

$$\check{g}_1 y^t g_0^{-1} e = (g_0 y \check{g}_1^{-1})^t e = \overline{g_0 y \check{g}_1^{-1} e} = \overline{\check{g}_2 y e} = g_2 \overline{y e} = g_2 y^t e,$$

establishing the second claim. □

By virtue of this theorem we may now describe $Spin(8)$ as

$$Spin(8) = \{(g_0, g_1, g_2) \text{ a } \theta\text{-triad of } SO(8)\}.$$

The standard orthogonal action of $Spin(8)$ on $S^7$ is given by

$$(g_0, g_1, g_2)x = \check{g}_2 \nu(x) e = \check{g}_2 x = \overline{g_2 \overline{x}}$$

for all $x \in S^7$. Recall that we embedded $\mathbf{Y}$ in ${}^2\mathbf{R}(8)$ by

$$y \mapsto \begin{pmatrix} y & 0 \\ 0 & y^t \end{pmatrix};$$

we may thus embed $x \in S^7$ in ${}^2\mathbf{R}(8)$ by

$$x \mapsto \begin{pmatrix} \nu(x) & 0 \\ 0 & \nu(x)^t \end{pmatrix}.$$

We claim that, for any $x \in S^7$, this embedding gives an element of $Spin(8)$. This is an immediate consequence of Lemma 2.11 and the following result.

**Lemma 2.12** *For any $x \in S^7$, the triple $\left(\nu(x)^t(\nu(x))\check{}, \nu(x), (\nu(x)^t)\check{}\right)$ is a $\theta$-triad of $SO(8)$.*

The proof of this lemma appears in Section 2.4.4.

We now relate the group of automorphisms of $\mathbf{O}$ to $Spin(8)$. Let

$$G_2 = \{g \in SO(8) : g(xy) = g(x)g(y) \text{ for all } x, y \in \mathbf{O}\}$$

denote the group of automorphisms of $\mathbf{O}$. If $g \in G_2$, then $g(e) = e$, so $g$ is of the form $\begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix}$ where $g' \in SO(7)$ and $\check{g} = g$.

**Lemma 2.13** *Let $g \in SO(8)$. Then $(g,g,g)$ is a $\theta$-triad of $SO(8)$ if and only if $g \in G_2$.*

**Proof** Suppose $g \in G_2$. If $y, z \in \mathbf{Y}$ are arbitrary, then

$$\begin{aligned} g((ye)(ze)) = g(ye)g(ze) &\Leftrightarrow gy\breve{g}^{-1}gze = (\breve{g}ye)(gze) \\ &\Leftrightarrow gx\breve{g}^{-1} \in \mathbf{Y} \text{ and } gx\breve{g}^{-1}e = \breve{g}xe \text{ for all } x \in \mathbf{Y}, \end{aligned}$$

so $(g,g,g)$ is a $\theta$-triad. Conversely, if $(g,g,g)$ is a $\theta$-triad of $SO(8)$, then for any $y \in \mathbf{Y}$, we have $gy\breve{g}^{-1} \in \mathbf{Y}$ and $gy\breve{g}^{-1}e = \breve{g}ye$. Set $y = I \in \mathbf{Y}$; then $he = e$, where $h = \breve{g}^{-1}g\breve{g}^{-1}$. It follows that $\check{h} = h$; this may be rewritten as $(g_1)^3 = I$, where $g_1 = g\breve{g}^{-1} \in \mathbf{Y}$. By Lemma 2.11, $(\breve{g}^{-1}, \breve{g}^{-1}, \breve{g}^{-1})$ is a $\theta$-triad of $SO(8)$, hence $(g_1, g_1, g_1)$ is also a $\theta$-triad. That is, $g_1 y\breve{g}_1^{-1}e = \breve{g}_1 ye$ for all $y \in \mathbf{Y}$. Since $\breve{g}_1 = g_1^{-1}$ and $(g_1)^3 = I$, this simplifies to $g_1ye = yg_1e$, or $(g_1e)(ye) = (ye)(g_1e)$, for all $y \in \mathbf{Y}$. It follows that $g_1e \in \mathbf{R}e$, so by the orthogonality of $g_1$ and the fact that $g_1 \in \mathbf{Y}$, we have $g_1 = \pm I$. It is easy to verify that $(-I,-I,-I)$ is not a $\theta$-triad of $SO(8)$, implying that $g_1 = I$ and $g = \breve{g}$. Consequently $gyg^{-1} \in \mathbf{Y}$ and $gyg^{-1}e = gye$ for all $y \in \mathbf{Y}$. Now if $x, z \in \mathbf{O}$, then let $x' = \nu(x)$, $z' = \nu(z)$; we have

$$g(xz) = g(x'z'e) = gx'g^{-1}gz'e = (gx'g^{-1}e)(gz'e) = (gx'e)(gz'e) = g(x)g(z),$$

that is, $g \in G_2$. □

For $i = 0,1,2$, define $H_i = \{(g_0, g_1, g_2) \in Spin(8) : g_ie = e\}$. Clearly $H_2$ is the stabiliser subgroup at $e \in S^7$ of the standard orthogonal action of $Spin(8)$, hence $H_2 \cong Spin(7)$. Furthermore, the automorphism $\theta : Spin(8) \to Spin(8)$ permutes the groups $H_i$ cyclically, so $H_i \cong Spin(7)$ for $i = 0,1,2$. If $(g_0, g_1, g_2) \in H_1 \cap H_2$, then $g_2e = e$ implies that $\breve{g}_2e = e$, whence $g_0\breve{g}_1^{-1} = I$ and $g_0 = \breve{g}_1$. Similarly, $g_1e = e$ implies that $g_2 = \breve{g}_0$. It follows that $(g_0, g_1, g_2) = (g_0, \breve{g}_0, \breve{g}_0)$ is a $\theta$-triad, whence so is $(g_0, \breve{g}_0, g_0)$ by Lemma 2.11. We have shown that $g_0 = \breve{g}_0$, so $(g_0, g_0, g_0)$ is a $\theta$-triad. By Lemma 2.13, $H_1 \cap H_2 = G_2$. Since $G_2$ is preserved by $\theta$, we see that $H_0 \cap H_1 = H_1 \cap H_2 = H_2 \cap H_0 = G_2$.

### 2.4.4 Further Remarks on Triality

In this section we link the two triality automorphisms $\Theta$ and $\theta$ of $\mathfrak{so}(8)$ and $Spin(8)$ respectively, present an equivalent characterisation of $\theta$-triads and use this characterisation to give a proof of Lemma 2.12.

Let $Z$ denote the centre of $Spin(8)$. It is easy to see that $Z$ is given by

$$Z = \{(I,I,I),(I,-I,-I),(-I,I,-I),(-I,-I,I)\} \cong \mathbf{Z}_2 \times \mathbf{Z}_2.$$

The triality automorphism $\theta$ preserves $(I,I,I)$ and cyclically permutes the other three elements of $Z$. It follows that $\theta$ projects to an automorphism of the space $Spin(8)/Z$ of order 3. The map $\Theta$ is the derivative of this automorphism. (Note that $\mathfrak{so}(8)$ may be regarded as the Lie algebra of $Spin(8)/Z$.) In fact $\theta$ does not project to an automorphism of $SO(8)$, for we have

$$\theta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \theta\begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

The arguments project to the same element of $SO(8)$ (namely $I$) but their images do not.

The group $Spin(8)$ is the only one of the groups $Spin(n)$ to admit a triality automorphism. That this is true is indicated by our method of construction using the threefold rotational symmetry of the Dynkin diagram for $\mathfrak{so}(8)$. The Dynkin diagrams for other Lie algebras $\mathfrak{so}(n)$, $n \neq 8$ do not possess this type of symmetry. Details may be found in [L].

The standard definition of a $\theta$-triad is a triple $(g_0, g_1, g_2)$ of elements of $SO(8)$ satisfying

$$\overline{g_0(\overline{xy})} = g_1(x)g_2(y)$$

for all $x, y \in \mathbf{O}$, or equivalently

$$g_0(xy) = \check{g}_2(x)\check{g}_1(y)$$

for all $x, y \in \mathbf{O}$. We show that this definition is equivalent to the one given in Section 2.4.3. Suppose that $(g_0, g_1, g_2)$ satisfies $g_0 k \check{g}_1^{-1} \in \mathbf{Y}$ for all $k \in \mathbf{Y}$ with $g_0 k \check{g}_1^{-1} e = \check{g}_2 k e$ for all $k \in \mathbf{Y}$. Fix $x, y \in \mathbf{O}$ and set $k = \nu(x) \in \mathbf{Y}$. There exists $z \in \mathbf{O}$ such that $g_0 \nu(x) \check{g}_1^{-1} = \nu(z)$. We have

$$\check{g}_2 x = \check{g}_2 \nu(x) e = g_0 \nu(x) \check{g}_1^{-1} e = \nu(z) e = z$$

and

$$g_0(xy) = g_0 \nu(x) y = \nu(z) \check{g}_1 y = \check{g}_2(x) \check{g}_1(y)$$

as required. Conversely, suppose that $g_0(xy) = \breve{g}_2(x)\breve{g}_1(y)$ for all $x, y \in \mathbf{O}$. Fixing $x \in \mathbf{O}$ and setting $z = \breve{g}_2(x)$ gives

$$g_0\nu(x)y = \nu(z)\breve{g}_1 y$$

for all $y \in \mathbf{O}$, whence $g_0\nu(x)\breve{g}_1^{-1} = \nu(z) \in \mathbf{Y}$ and $g_0\nu(x)\breve{g}_1^{-1}e = z = \breve{g}_2(x)$ as required.

**Proof of Lemma 2.12** We use the standard definition of a $\theta$-triad. The map $\nu(x)$ corresponds to left multiplication by $x \in \mathbf{O}$, whereas the map $(\nu(x))\check{}$ corresponds to right multiplication by $\overline{x}$, since

$$(\nu(x))\check{}\,a = \overline{x\overline{a}} = a\overline{x}$$

for all $a \in \mathbf{O}$. Consider the triad $(g_0, g_1, g_2) = \Big(\nu(x)^t(\nu(x))\check{}, \nu(x), (\nu(x)^t)\check{}\Big)$ for some $x \in S^7$. For any $a, b \in \mathbf{O}$, we have

$$\begin{aligned}
\overline{g_0(\overline{ab})} &= \overline{\overline{x}((\overline{ab})\overline{x})} \\
&= (x(ab))x \\
&= (xa)(bx) \\
&= g_1(a)g_2(b)
\end{aligned}$$

by the Moufang identity (Lemma 2.10 (iii)). Since $g_0, g_1, g_2 \in SO(8)$, the result follows. $\square$

# Chapter 3

# The Construction of Symmetric Spaces

In order to formulate the new construction of symmetric spaces of rank one of non-compact type, we first require some properties of a special class of Lie algebras known as $H$-type algebras, introduced by Kaplan [K]. Of particular interest is the subclass of these algebras satisfying a condition known as the $J^2$ condition, a condition first described in [CDKR]. We use these algebras in conjunction with Clifford algebras to construct the symmetric spaces, examining each of the four families individually. We also examine a result due to Pansu [Pu] involving graded isomorphisms of $H$-type algebras.

## 3.1 $H$-type algebras

In this section we define $H$-type algebras and give some of their important properties. We also examine some associated Riemannian manifolds, generalising the Klein and upper half-space models of hyperbolic space. Most of the material in this section appears in [CDKR2], although some of the results were originally proved in [K].

### 3.1.1 The $J^2$ condition

Let $\mathfrak{n}$ denote a Lie algebra equipped with an inner product $\langle \cdot , \cdot \rangle$ and associated length $|\cdot|$. Suppose that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with $[\mathfrak{n}, \mathfrak{z}] = \{0\}$ and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}$, so that $\mathfrak{n}$ is two-step nilpotent. Define the map $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ by

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle \tag{3.1}$$

for all $Z \in \mathfrak{z}$ and all $X, Y \in \mathfrak{v}$. If $J$ satisfies

$$|J_Z X| = |Z|\,|X| \tag{3.2}$$

for all $Z \in \mathfrak{z}$, $X \in \mathfrak{v}$ then $\mathfrak{n}$ is said to be of *Heisenberg type*, or alternatively $\mathfrak{n}$ is said to be an *H-type algebra*. The associated simply connected Lie group $N = \exp(\mathfrak{n})$ is called an *H-type group*.

The map $J$ possesses several important properties. Equation 3.1 implies that $J_Z$ is skew-symmetric for all $Z \in \mathfrak{z}$. Repeated polarisation of (3.2) gives

$$\langle J_Z X, J_Z Y\rangle = \langle Z, Z\rangle \langle X, Y\rangle \tag{3.3}$$

$$\langle J_W X, J_Z X\rangle = \langle W, Z\rangle \langle X, X\rangle \tag{3.4}$$

$$\langle J_W X, J_Z Y\rangle + \langle J_Z X, J_W Y\rangle = 2\langle W, Z\rangle \langle X, Y\rangle \tag{3.5}$$

for all $X, Y \in \mathfrak{v}$, $W, Z \in \mathfrak{z}$. By the skew-symmetry of $J_Z$ and (3.3) we have

$$J_Z^2 = -|Z|^2 I \tag{3.6}$$

hence

$$J_Z J_W + J_W J_Z = -2\langle Z, W\rangle I \tag{3.7}$$

by polarisation, for all $Z, W \in \mathfrak{z}$. This implies that when $Z$ and $W$ are orthogonal,

$$J_Z J_W = -J_W J_Z. \tag{3.8}$$

By (3.1) and (3.4),

$$\langle W, [X, J_Z X]\rangle = \langle J_W X, J_Z X\rangle = \langle W, Z\rangle \langle X, X\rangle$$

for all $X \in \mathfrak{v}$, $W, Z \in \mathfrak{z}$, implying that

$$[X, J_Z X] = |X|^2 Z \tag{3.9}$$

for all $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$. Finally, for any $X, Y \in \mathfrak{v}$, $W, Z \in \mathfrak{z}$, we have

$$\begin{aligned}
\langle W, [J_Z X, J_Z Y]\rangle &= \langle J_W J_Z X, J_Z Y\rangle \\
&= -\langle J_Z J_W X, J_Z Y\rangle - 2\langle W, Z\rangle \langle X, J_Z Y\rangle \\
&= -\langle Z, Z\rangle \langle J_W X, Y\rangle + 2\langle W, Z\rangle \langle J_Z X, Y\rangle \\
&= -\left\langle W, \left(|Z|^2 [X, Y] - 2Z\langle Z, [X, Y]\rangle\right)\right\rangle
\end{aligned}$$

by (3.1), (3.7), (3.3), the skew-symmetry of $J_Z$ and (3.1) again, thus

$$[J_Z X, J_Z Y] = -|Z|^2 \rho_Z[X,Y] \tag{3.10}$$

for all $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, where $\rho_Z : \mathfrak{z} \to \mathfrak{z}$ is the reflection in the hyperplane orthogonal to $Z$.

Given $X \in \mathfrak{v}\backslash\{0\}$, we let $\mathfrak{j}(X) = \{J_Z X : Z \in \mathfrak{z}\}$.

**Definition** Given an $H$-type algebra $\mathfrak{n}$, we say that $\mathfrak{n}$ satisfies the $J^2$ *condition* if for all $X \in \mathfrak{v}$, $Z_1, Z_2 \in \mathfrak{z}\backslash\{0\}$ with $\langle Z_1, Z_2\rangle = 0$, there exists $Z_3 \in \mathfrak{z}$ such that

$$J_{Z_1} J_{Z_2} X = J_{Z_3} X. \tag{3.11}$$

Equivalently, $\mathfrak{n}$ satisfies the $J^2$ condition if $J_Z$ preserves $\mathbf{R}X \oplus \mathfrak{j}(X)$ for all $Z \in \mathfrak{z}$, $X \in \mathfrak{v}$, or if

$$(\mathbf{R} + \mathfrak{j})(\mathbf{R} + \mathfrak{j})X = (\mathbf{R} + \mathfrak{j})X$$

for all $X \in \mathfrak{v}$. By (3.2) and (3.8), if $X \neq 0$ and $\dim(\mathfrak{z}) \geq 2$, then the element $Z_3$ in (3.11) is unique and orthogonal to both $Z_1, Z_2$.

Suppose that $\dim(\mathfrak{z}) = q > 0$ and let $\{Z_i\}_{i=1}^q$ be an orthonormal basis for $\mathfrak{z}$. The map $C(q) \to \mathrm{End}(\mathfrak{v})$; $e_i \mapsto J_{Z_i}$ extends by linearity to a representation of $C(q)$ on $\mathfrak{v}$. Now let $\mathfrak{n}$ satisfy the $J^2$ condition. If $X \in \mathfrak{v}\backslash\{0\}$ and $X' \in (\mathbf{R}X \oplus \mathfrak{j}(X))^\perp$, then the representations of $C(q)$ on the mutually orthogonal $J_\mathfrak{z}$-invariant subspaces $\mathbf{R}X \oplus \mathfrak{j}(X)$ and $\mathbf{R}X' \oplus \mathfrak{j}(X')$ are equivalent. To see this, suppose for some $Z, Z', W, W' \in Z$ we have $\langle Z, Z'\rangle = 0$, $J_Z J_{Z'} X = J_W X$ and $J_Z J_{Z'} X' = J_{W'} X'$. By the $J^2$ condition, there exists $W_0 \in \mathfrak{z}$ with $J_Z J_{Z'}(X + X') = J_{W_0}(X + X')$, implying that $J_Z J_{Z'} X = J_{W_0} X$ and $J_Z J_{Z'} X' = J_{W_0} X'$, that is, $W = W_0 = W'$. Consequently the linear map that sends $X$ to $X'$ and $J_Z X$ to $J_Z X'$ for each $Z \in \mathfrak{z}$ is an intertwining operator as required. It follows that

$$\mathfrak{v} = \bigoplus_{k=1}^{m} (\mathbf{R}X_k \oplus \mathfrak{j}(X_k))$$

for some $X_1, \ldots, X_k \in \mathfrak{v}$, where each summand is an equivalent irreducible Clifford submodule.

### 3.1.2 Some Associated Riemannian Spaces

Suppose $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is an $H$-type algebra satisfying the $J^2$ condition. Let $\mathfrak{a}$ denote a one-dimensional normed vector space with unit vector $H$ and let $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$. In order to make $\mathfrak{s}$ into a Lie algebra with inner product, we extend the inner product on $\mathfrak{n}$ by requiring that $\mathfrak{a}$ be orthogonal to $\mathfrak{n}$; we extend the Lie bracket by bilinearity and the conditions

$$[H, X] = \frac{1}{2}X \qquad \text{and} \qquad [H, Z] = Z \tag{3.12}$$

for all $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$. We write $(X, Z, t)$ for the element $X + Z + tH$ of $\mathfrak{s}$, where $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$, $t \in \mathbf{R}$. Define the *height function* $h : \mathfrak{s} \to \mathbf{R}$ by

$$h(X, Z, t) = t - \frac{1}{4}|X|^2$$

for $(X, Z, t) \in \mathfrak{s}$, and let

$$D = \{p \in \mathfrak{s} : h(p) > 0\}.$$

The space $D$ is the analogue of the upper half-plane model of hyperbolic space. It also generalises the well-known Siegel domain for $SU(2,1)$. Let $\exp(\mathfrak{s})$ denote the connected, simply connected Lie group with Lie algebra $\mathfrak{s}$. This group may be identified with $S = \mathfrak{v} \times \mathfrak{z} \times \mathbf{R}^+$ by identifying the point $\exp(X + Z)\exp(\log tH)$ of $\exp(\mathfrak{s})$ with $(X, Z, t) \in S$. Define $\Theta : S \to S$ by

$$\Theta(X, Z, t) = \left(X, Z, t + \frac{1}{4}|X|^2\right) \tag{3.13}$$

for all $(X, Z, t) \in \mathfrak{s}$. It is trivial to see that $\Theta$ is injective and $\Theta(S) = D$. It follows that there exists a simply transitive action of $\exp(\mathfrak{s})$ on $D$ given by conjugating left multiplication in the group $S$ by $\Theta$. We obtain an invariant metric on $D$ by transporting the left-invariant metric of $\exp(\mathfrak{s})$ to $D$, requiring that $\Theta$ be an isometry. We now define the *Cayley transform* $C : B \to D$, where $B$ is the unit ball in $\mathfrak{s}$, by

$$C(X, Z, t) = \frac{1}{(1-t)^2 + |Z|^2}\left(2(1 - t + J_Z)X, 2Z, 1 - t^2 - |Z|^2\right) \tag{3.14}$$

for all $(X, Z, t) \in B$. The inverse of $C$ is given by

$$C^{-1}(X', Z', t') = \frac{1}{(1+t')^2 + |Z'|^2}\left((1 + t' - J_{Z'})X', 2Z', -1 + (t')^2 + |Z'|^2\right) \tag{3.15}$$

for all $(X', Z', t') \in D$. We obtain a metric on $B$ by transporting the metric on $D$ to $B$, requiring that $C$ be an isometry. Note that we may identify the tangent space at a point $p$ of either $D$ or $B$ with $\mathfrak{s}$ since $D$ and $B$ are subsets of the vector space

$\mathfrak{s}$. Any isometry $g$ of $D$ may be transported using $C$ to give an isometry $\tilde{g}$ of $B$: explicitly, $\tilde{g} = C^{-1}gC$.

Define the *inversion* $\sigma : D \to D$ by

$$\sigma(X, Z, t) = \frac{1}{|Z|^2 + t^2}((-t + J_Z)X, -Z, t) \tag{3.16}$$

for all $(X, Z, t) \in D$. The equivalent map on the ball $B$ is given by

$$\tilde{\sigma}(X, Z, t) = C^{-1}\sigma C(X, Z, t) = -(X, Z, t)$$

for all $(X, Z, t) \in B$. It is proved in [CDKR2] that $\sigma$ is an isometry if and only if $\mathfrak{n}$ satisfies the $J^2$ condition. It is also demonstrated that the metric on $B$ is given by

$$\langle v, v \rangle_p = \begin{cases} \dfrac{4\,|v_{\mathrm{rad}}|^2}{\left(1 - |p|^2\right)^2} + \dfrac{4\,|v_{\mathrm{tan}}|^2}{1 - |p|^2}, & p \neq 0 \\ 4\,|v|^2, & p = 0 \end{cases} \tag{3.17}$$

where

$$v_{\mathrm{rad}} = P_{T_p^{(2)} \oplus \mathbf{R}p} v$$

$$v_{\mathrm{tan}} = v - v_{\mathrm{rad}}$$

and for $p = (X, Z, t) \neq 0$,

$$T_p^{(2)} \oplus \mathbf{R}p = \mathbf{R}X + J_{\mathfrak{z}}X \tag{3.18}$$

if $Z = t = 0$, and

$$T_p^{(2)} \oplus \mathbf{R}p = \{((u + J_W)(t - J_Z)X, (|Z|^2 + t^2)W, (|Z|^2 + t^2)u) : W \in \mathfrak{z}, u \in \mathbf{R}\} \tag{3.19}$$

otherwise. Here $P$ denotes orthogonal projection.

Let $N$ and $A$ denote the subgroups $\exp(\mathfrak{n})$ and $\exp(\mathfrak{a})$ of $S$. It is easy to see that $N$ is normal in $S$ and $S$ is the semidirect product of $N$ with $A$. We may write $A = \{a_u\}_{u \in \mathbf{R}^+}$ where $a_u = \exp(\log uH)$ acts on $D$ by

$$a_u(X, Z, t) = (u^{1/2}X, uZ, ut)$$

for all $(X, Z, t) \in D$. A messy computation shows that the action of $a_u$ on $B$ is given by

$$\tilde{a}_u(X, Z, t) = m^{-1}((s(t - J_Z) + c)X, Z, cs(1 + t^2 + |Z|^2) + (c^2 + s^2)t)$$

where

$$c = \frac{u+1}{2\sqrt{u}}, \qquad s = \frac{u-1}{2\sqrt{u}}, \qquad m = |s(Z \oplus tH) + cH|^2 = (ts+c)^2 + s^2 |Z|^2.$$

Note that $c^2 - s^2 = 1$.

Let $G$ denote the group of isometries of $B$ and $K$ be the stabiliser subgroup of the origin $(0,0,0) \in B$. Let $L$ denote the subgroup of $K$ consisting of isometries which preserve $\mathfrak{v}$ (hence $\mathfrak{z} \oplus \mathfrak{a}$) and let $M$ denote the subgroup of $L$ consisting of isometries which fix $H$. Then $K$ is a group of orthogonal transformations, $K$ acts transitively on $S_{\mathfrak{s}}$ and $M$ acts transitively on $S_{\mathfrak{v}} \times S_{\mathfrak{z}}$, where $S_{\mathfrak{s}}, S_{\mathfrak{v}}, S_{\mathfrak{z}}$ are the unit spheres in $\mathfrak{s}, \mathfrak{v}, \mathfrak{z}$ respectively. (Proofs may be found in [CDKR2].)

## 3.2 Construction of Symmetric Spaces Using H-type Algebras

In this section we present our construction of all symmetric spaces of rank one of noncompact type. The construction utilises $H$-type algebras, Clifford algebras and Spin groups. We also identify the subgroups $K$, $L$ and $M$ of isometries of the symmetric spaces.

Let $\mathfrak{v}$ denote a nontrivial Clifford module for $C(q)$, $q > 0$, with a compatible inner product. Let $\mathfrak{z} = Q \subset C(q)$ and $\mathfrak{a} = \mathrm{span}\{1\} \subset C(q)$. Let $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ where the inner product is extended by linearity and orthogonality. An element $g \in Spin(q+1) \subset C(q)$ acts on $\mathfrak{v}$ by the Clifford action $X \mapsto \pi_C(g)X = gX$ and on $\mathfrak{z} \oplus \mathfrak{a} = Q'$ by the orthogonal action $Z \mapsto \pi_O(g)Z = gZ\widehat{g}^{-1}$. The unit elements of $Q'$ may be regarded as a copy of $S^q$ inside $Spin(q+1)$. For any unit $Z \in \mathfrak{z}$ and $X \in \mathfrak{v}$ we define $J_Z X = \pi_C(Z)X$ and extend $J$ to all of $\mathfrak{z}$ by linearity. The Lie bracket is determined by (3.1) and (3.12). Having equipped the unit ball of $\mathfrak{s}$ with the metric of (3.17), we seek descriptions of the subgroups $M, L, K$ of the group of isometries $G$ of $\mathfrak{s}$ in the case where $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ satisfies the $J^2$ condition. In a series of results following this discussion, we show that the group $L$ is given by

$$L = \{(g, \theta) \in Spin(q+1) \times O(\mathfrak{v}) : \theta J_Z = \varepsilon J_Z \theta \text{ for all } z \in Z, \text{ for some } \varepsilon = \pm 1\}$$

where $(g, \theta)$ acts on $\mathfrak{s}$ by

$$(X, Z \oplus tH) \mapsto (\theta \pi_C(g)X, \pi_O(g)(Z \oplus tH))$$

for all $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$, $t \in \mathbf{R}$. Note that the decomposition of an element of $L$ into the form $(g, \theta)$ need not be unique and that if $\varepsilon = 1$ then $\theta$ is an intertwining operator for $\pi_C$. In Sections 3.3 and 3.4 we shall see that if $\dim \mathfrak{z} \equiv 3 \pmod 4$ then $\varepsilon = 1$ is the only possibility.

We also demonstrate that the subgroup $M$ of $L$ is given by

$$M = \{(g, \theta) \in L : g \in Spin(q)\}$$

where $Spin(q)$ is embedded in $Spin(q+1)$ as described in Section 2.3.4.

A pair $(\lambda, Y)$ where $\lambda \in \mathbf{R}$ and $Y \in \mathfrak{v}$ acts on $\mathfrak{s}$ by

$$(X, Z, t) \mapsto ((t - \lambda u)Y + J_{(Z - \lambda W)}Y + X_2, \lambda Z + W, \lambda t + u),$$

where $X = uY \oplus J_W Y \oplus X_2$ with $u \in \mathbf{R}$, $W \in \mathfrak{z}$ and $X_2 \in (\mathbf{R}Y \oplus \mathfrak{j}(Y))^\perp$. We show that the group $K$ is then given by

$$K = \left\{((\lambda, Y), l) \in (\mathbf{R} \times \mathfrak{v}) \times L : \lambda^2 + |Y|^2 = 1\right\}$$

where the element $((\lambda, Y), l)$ acts on $v \in \mathfrak{s}$ by

$$v \mapsto (\lambda, Y) l v.$$

The full group of isometries $G$ is then given by the Cartan decomposition

$$G = KAK$$

or the Iwasawa decomposition

$$G = NAK.$$

**Definition** A *graded automorphism* of $\mathfrak{n}$ is a Lie algebra automorphism of $\mathfrak{n}$ which preserves $\mathfrak{v}$ and $\mathfrak{z}$. Equivalently, a graded automorphism is a pair $(A, B)$ where $A \in GL(\mathfrak{v})$ and $B \in GL(\mathfrak{z})$ such that

$$[AX, AY] = B[X, Y]$$

for all $X, Y \in \mathfrak{v}$.

**Theorem 3.1**

*(i) If $m \in M$ acts on $\mathfrak{s}$ by $m(X,Z,t) = (AX, BZ, t)$ for all $(X,Z,t) \in M$, where $A \in O(\mathfrak{v})$, $B \in O(\mathfrak{z})$, then $m$ restricts to a graded automorphism of $\mathfrak{n}$. Conversely, any graded automorphism $(A,B)$ of $\mathfrak{n}$ with $A$ and $B$ orthogonal extends to an element of $M$.*

*(ii) If $(A,B)$ is a graded automorphism of $\mathfrak{n}$, where $A$ and $B$ are orthogonal, then $B = \pm\pi_O(g)|_{\mathfrak{z}}$ for some $g \in Spin(q)$ and $A = \pi_C(g)\theta$ where $\theta \in O(\mathfrak{v})$ satisfies $\theta J_Z = \det(B) J_Z \theta$ for all $Z \in \mathfrak{z}$.*

**Proof** (i) Let $m = (A,B,I)$ on $\mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$. Fix $p = (X,-Z,0) \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$. Since $m$ is a linear isometry, we have $m(T_p^{(2)} \oplus \mathbf{R}p) = T_{mp}^{(2)} \oplus \mathbf{R}mp$. Setting

$$v = \left(J_Z X, 0, |Z|^2\right) \in T_p^{(2)} \oplus \mathbf{R}p,$$

we must have

$$mv = \left(AJ_Z X, 0, |Z|^2\right) = \left((u + J_W) J_{BZ} AX, |BZ|^2 W, |BZ|^2 u\right)$$

for some $u \in \mathbf{R}$, $W \in \mathfrak{z}$. Then $W = 0$, $u = 1$ (as $B$ is orthogonal), so

$$AJ_Z X = J_{BZ} AX$$

for all $X \in \mathfrak{v}$, $Z \in \mathfrak{z}$. This implies that

$$\begin{aligned} \langle Z, B^{-1}[AX, AY]\rangle &= \langle BZ, [AX, AY]\rangle = \langle J_{BZ} AX, AY\rangle \\ &= \langle AJ_Z X, AY\rangle = \langle J_Z X, Y\rangle = \langle Z, [X,Y]\rangle \end{aligned}$$

for all $X, Y \in \mathfrak{v}$, $Z \in \mathfrak{z}$, by the orthogonality of $A$ and $B$. It follows that

$$[AX, AY] = B[X,Y]$$

for all $X, Y \in \mathfrak{v}$, as required.

Conversely, suppose that $(A,B)$ is a graded automorphism of $\mathfrak{n}$. For all $X, Y \in \mathfrak{v}$, $Z \in \mathfrak{z}$,

$$\begin{aligned} \langle J_{BZ} AX, AY\rangle &= \langle BZ, [AX, AY]\rangle = \langle BZ, B[X,Y]\rangle \\ &= \langle Z, [X,Y]\rangle = \langle J_Z X, Y\rangle = \langle AJ_Z X, AY\rangle, \end{aligned}$$

that is,

$$AJ_Z = J_{BZ}A \tag{3.20}$$

for all $Z \in \mathfrak{z}$.

We need to show that if $m = (A, B, I)$, then $m\left(T_p^{(2)} \oplus \mathbf{R}p\right) = T_{mp}^{(2)} \oplus \mathbf{R}mp$ for all $p \in \mathfrak{s}$. Let $p = (X, Z, t)$. If $Z = 0$ and $t = 0$, then $mp = (AX, 0, 0)$, so

$$m\left(T_p^{(2)} \oplus \mathbf{R}p\right) = \{(A(uX + J_W X), 0, 0) : u \in \mathbf{R}, W \in \mathfrak{z}\}$$

whereas

$$T_{mp}^{(2)} \oplus \mathbf{R}mp = \{(u'AX + J_{W'}AX, 0, 0) : u \in \mathbf{R}, W' \in \mathfrak{z}\}.$$

If $u = u'$ and $W' = BW$, then

$$A(uX + J_W X) = u'AX + J_{W'}AX$$

by (3.20), so equality holds in this case. If $(Z, t) \neq 0$, then

$$\begin{aligned} &m\left(T_p^{(2)} \oplus \mathbf{R}p\right) \\ &\quad = \left\{\left(A(u + J_W)(t - J_Z)X, \left(|Z|^2 + t^2\right) BW, \left(|Z|^2 + t^2\right) u\right) : u \in \mathbf{R}, W \in \mathfrak{z}\right\} \end{aligned}$$

while

$$\begin{aligned} &T_{mp}^{(2)} \oplus \mathbf{R}mp \\ &\quad = \left\{((u' + J_{W'})(t - J_{BZ})AX, \left(|Z|^2 + t^2\right) W', \left(|Z|^2 + t^2\right) u') : u' \in \mathbf{R}, W' \in \mathfrak{z}\right\}. \end{aligned}$$

Now if $u' = u$ and $W' = BW$ then

$$\begin{aligned} (u' + J_{W'})(t - J_{BZ})AX &= (u + J_{BW})(t - J_{BZ})AX \\ &= A(u + J_W)(t - J_Z)X \end{aligned}$$

by (3.20), establishing the result.

(ii) We have $AJ_W = J_{BW}A$ for all $W \in \mathfrak{z}$. We may write

$$B = b(-\rho_{Z_1}) \cdots (-\rho_{Z_m})$$

for some unit vectors $Z_1, \ldots, Z_m \in \mathfrak{z}$, where $b = \det(B) = \pm 1$. Write

$$A = J_{Z_1} \cdots J_{Z_m} \theta$$

where $\theta$ is orthogonal.

If $Z, Z' \in \mathfrak{z}$, then

$$J_{\rho_{Z'}Z}J_{Z'} = \left(J_Z - 2\frac{\langle Z', Z\rangle}{|Z'|^2}J_{Z'}\right)J_{Z'} = J_ZJ_{Z'} + 2\langle Z', Z\rangle I = -J_{Z'}J_Z \tag{3.21}$$

by (3.7). It follows that for all $W \in \mathfrak{z}$,

$$\begin{aligned}J_{BW}A &= bJ_{(-\rho_{Z_1})\cdots(-\rho_{Z_m})W}J_{Z_1}\cdots J_{Z_m}\theta\\ &= bJ_{Z_1}J_{(-\rho_{Z_2})\cdots(-\rho_{Z_m})W}J_{Z_2}\cdots J_{Z_m}\theta\\ &= \dots\\ &= bJ_{Z_1}\cdots J_{Z_m}J_W\theta\end{aligned}$$

whereas

$$J_{BW}A = AJ_W = J_{Z_1}\cdots J_{Z_m}\theta J_W,$$

implying that

$$\theta J_W = bJ_W\theta.$$

Since $\dim\mathfrak{z}$ is odd, we may express $B$ as

$$B = b\rho_{W_1}\cdots\rho_{W_k}$$

for some $W_1, \dots, W_k$ with $k$ even. The result follows from the observations that for all odd $i$ less than $k$, $\pi_O(W_iW_{i+1}) = \rho_{W_i}\rho_{W_{i+1}}$, $\pi_C(W_iW_{i+1}) = J_{W_i}J_{W_{i+1}}$ and $W_iW_{i+1} \in Spin(q)$. □

Theorem 3.1 immediately implies that

$$M = \{(g,\theta) \in Spin(q) \times O(\mathfrak{v}) : \theta J_Z = \varepsilon J_Z\theta \text{ for all } Z \in \mathfrak{z}, \text{ for some } \varepsilon = \pm 1\}.$$

**Theorem 3.2** *Any element of $L$ is of the form $(g,\theta)$ where $g \in Spin(q+1)$ and $\theta \in O(\mathfrak{v})$ satisfies $\theta J_Z = \varepsilon J_Z\theta$ for some $\varepsilon = \pm 1$ and all $Z \in \mathfrak{z}$.*

**Proof** We show that $(Z, I) \in L$ where $Z \in S^q$ and $I$ is the identity. If $p \in \mathfrak{s}$, we may write $p = (X, Y)$ where $X \in \mathfrak{v}$, $Y \in \mathfrak{z} \oplus \mathfrak{a} = Q'$. In the same manner we may write

$$T_p^{(2)} \oplus \mathbf{R}p = \{(\pi_C(W)\pi_C(\overline{Y})X, |Y|^2 W) : W \in Q'\}$$

if $Y \neq 0$ and

$$T_p^{(2)} \oplus \mathbf{R}p = \pi_C(Q')X$$

otherwise. Now $(Z, I)$ acts on $p$ to give $(\pi_C(Z)X, \pi_O(Z)Y)$. We need to check that this action preserves the appropriate subspace, that is,

$$T^{(2)}_{(Z,I)p} \oplus \mathbf{R}(Z, I)p = (Z, I)(T^{(2)}_p \oplus \mathbf{R}p).$$

Since $Z \in S^q$, we have

$$|\pi_O(Z)Y| = |Y|$$

for all $Y \in \mathfrak{z} \oplus \mathfrak{a}$. It follows that we need only verify that

$$\pi_C(\pi_O(Z)W)\pi_C(\overline{\pi_O(Z)Y})\pi_C(Z) = \pi_C(Z)\pi_C(W)\pi_C(\overline{Y}) \tag{3.22}$$

for all $W \in \mathfrak{z} \oplus \mathfrak{a}$, and that

$$\pi_C(Q')\pi_C(Z)X = \pi_C(Z)\pi_C(Q')X. \tag{3.23}$$

In (3.22),

$$\begin{aligned} \text{LHS} &= \pi_C\left(ZW\widehat{Z}^{-1}\overline{ZY\widehat{Z}^{-1}Z}\right) \\ &= \pi_C(ZWZ\overline{ZY}ZZ) \\ &= \pi_C(ZW\bar{Y}) = \text{RHS} \end{aligned}$$

since $Z \in S^q$. Furthermore, (3.23) is true by the $J^2$ condition. It follows that $(Z, I) \in L$ as claimed. Now given $g \in L$, choose $Z \in S^q$ such that $((Z, I) \circ g)H = H$; this is always possible by the transitivity of the orthogonal action of $S^q$ on $Q'$. It follows that $(Z, I) \circ g$ is in $M$, so this composition may be expressed as $(g_0, \theta)$ for some $g_0 \in Spin(q)$ and some $\theta$ satisfying the requirements of the statement of the theorem. We have shown that $g = (Z^{-1}g_0, \theta)$ with $Z^{-1}g_0 \in Spin(q+1)$ as required. □

**Theorem 3.3** *Every element of $K$ is expressible in the form $((\lambda, Y), l) \in (\mathbf{R}\times\mathfrak{v})\times L$ with $\lambda^2 + |Y|^2 = 1$.*

**Proof** Let $k \in K$. By composing with an element of $L$, we may suppose that $k$ maps $(0, 0, 1)$ to $(X_0, 0, t_0)$ for some $X_0 \in \mathfrak{v}$, $t_0 \in \mathbf{R}$ with $t_0^2 + |X_0|^2 = 1$. Now

$$\begin{aligned} k(\mathfrak{z} \oplus \mathfrak{a}) &= k(T^{(2)}_H \oplus \mathbf{R}H) = T^{(2)}_{kH} \oplus \mathbf{R}kH \\ &= \{((u + J_W)X_0, t_0W, t_0u) : (W, u) \in \mathfrak{z} \oplus \mathfrak{a}\}. \end{aligned}$$

As in Section 4 of [CDKR2], let $\theta$ denote the differential of the involution $G \to G : g \mapsto \sigma g \sigma$ at the identity of $G$. (The map $\Theta$ should not be confused with the triality automorphism of Chapter 2.) The isometry $\exp\left(x(\theta Y + Y)\tilde{}\right)$, where $\cos(x) = t_0$, $\sin(x) = -|X_0|$ and $Y = X_0/|X_0|$, acts ([CDKR2] p. 31) in the same way as $(t_0, X_0)$ does. In fact

$$(t_0, X_0)(0, Z, t) = (tX_0 + J_Z X_0, t_0 Z, t_0 t),$$

so the isometry $(t_0, X_0)^{-1}k$ preserves $\mathfrak{z} \oplus \mathfrak{a}$ hence is in $L$. The result follows immediately. □

## 3.3 Applying the Construction

We now explain how the construction applies to each of the four families of symmetric spaces of rank one of noncompact type. In particular, we identify the groups $G, K, L, M$ for each of these spaces and describe their actions.

### 3.3.1 Case 1: $PO(1,n)/O(n)$

The cases when $\mathfrak{v} = 0$ or $\mathfrak{z} = 0$ correspond to the Poincaré and Klein models of real hyperbolic space respectively. In either case, the $J^2$ condition is trivially satisfied. Due to the degeneracy of the $H$-type algebras, the Clifford algebraic interpretation is largely irrelevant.

In the case where $\mathfrak{v} = 0$,

$$T_p^{(2)} \oplus \mathbf{R}p = \mathfrak{s}$$

for all $p \in \mathfrak{s}$, implying that $v_{\text{rad}} = v$ and $v_{\tan} = 0$ for all $v \in T_pB$. The metric is then

$$\langle v, v \rangle_p = \frac{4|v|^2}{\left(1 - |p|^2\right)^2}$$

for all $v \in T_pB$, which is precisely the Poincaré metric on the unit ball $B^n$ where $n = \dim(\mathfrak{s})$. In this case, we have $G \cong PO(1,n)$, $K = L = O(\mathfrak{s}) \cong O(n)$ and $M = O(\mathfrak{z}) \cong O(n-1)$.

Now suppose instead that $\mathfrak{z} = 0$. The map $J$ is trivial, so

$$T_p^{(2)} \oplus \mathbf{R}p = \mathbf{R}X \oplus \mathfrak{j}(X) = \mathbf{R}X = \mathbf{R}p$$

if $p = (X, 0, 0)$ and

$$T_p^{(2)} \oplus \mathbf{R}p = \{(utX, 0, t^2u) : u \in \mathbf{R}\} = \mathbf{R}p$$

if $p = (X, 0, t)$, $t \neq 0$. In either case, $v_{\text{rad}}$ is the projection of $v$ onto $p$ for any $v \in T_pB$. The metric is thus four times the ordinary Klein metric on the unit ball $B^n$ where $n = \dim(\mathfrak{s})$. In this case, $G \cong PO(1, n)$, $K = O(\mathfrak{s}) \cong O(n)$ and $L = M = O(\mathfrak{v}) \cong O(n-1)$.

### 3.3.2 Case 2: $PU(1, n)/U(n)$

Suppose $q = \dim(\mathfrak{z}) = 1$. The group $Spin(2) \cong U(1) \cong S^1$ acts orthogonally on $Q' \cong \mathbf{C}$ by

$$\pi_O(g)x = g^2x$$

for all $g \in S^1$, $x \in \mathbf{C}$. The Clifford action on $\mathfrak{v} = \mathbf{C}^{n-1}$ is given by

$$\pi_C(g)X = gX$$

for all $g \in S^1$, $X \in \mathbf{C}^{n-1}$. Identify $\mathfrak{z}$ with $\mathrm{Im}(\mathbf{C})$ and $\mathfrak{a}$ with $\mathrm{Re}(\mathbf{C})$. The sphere $S^1$ is the whole of $Spin(2)$. It follows that $J_iX = iX$ for all $X \in \mathfrak{v} = \mathbf{C}^{n-1}$. The $J^2$ condition holds trivially since there are no nonzero orthogonal elements $Z_1, Z_2 \in \mathfrak{z}$. Given $X, Y \in \mathfrak{v}$, we know that $[X, Y] = ki$ for some $k \in \mathbf{R}$, however

$$k = \mathrm{Re}(\bar{\imath}[X, Y]) = \langle i, [X, Y]\rangle = \langle iX, Y\rangle = \mathrm{Re}(\bar{\imath}(X, Y)_{\mathbf{C}}) = \mathrm{Im}(X, Y)_{\mathbf{C}}$$

where

$$(X, Y)_{\mathbf{C}} = \sum_{j=1}^{n-1} \overline{X}_j Y_j, \qquad \langle X, Y\rangle = \mathrm{Re}(X, Y)_{\mathbf{C}}, \qquad \langle Z_1, Z_2\rangle = \mathrm{Re}(\overline{Z}_1 Z_2)$$

for all $X, Y \in \mathfrak{v}$, $Z_1, Z_2 \in \mathfrak{z} \oplus \mathfrak{a}$. For any $p = (X, iZ + t) \in \mathbf{C}^n$ we have

$$T_p^{(2)} \oplus \mathbf{R}p = \left\{\left((u + iw)(t - iZ)X, \left(Z^2 + t^2\right)(iw + u)\right) : u, w \in \mathbf{R}\right\} = \mathbf{C}p,$$

thus the metric agrees with the one given in Section 1.3 up to a constant factor.

We seek orthogonal automorphisms of $\mathfrak{v}$ which commute or anticommute with $J_i$. Since conjugation anticommutes with $J_i$ and the only orthogonal maps which commute with $J_i$ are the elements of $U(1, n)$, we see that any element of $L$ is given by $(A_0, g_0, \varepsilon)$ with $A_0 \in U(n-1)$, $g_0 \in S^1$ and $\varepsilon = 0$ or 1, where

$$(A_0, g_0, \varepsilon)(X, Y) = (g_0 A_0 \sigma^{\varepsilon} X, g_0^2 \sigma^{\varepsilon} Y)$$

for all $(X,Y) \in \mathfrak{v} \times (\mathfrak{z} \oplus \mathfrak{a})$. (Here $\sigma$ is componentwise complex conjugation as in Chapter 1.) Replacing $g_0 A_0 \in U(n-1)$ by $A_1$ and $g_0^2 \in S^1$ by $g_1$, we see that $L \cong U(n-1) \times U(1) \times \mathbf{Z}_2$ with $(A_1, g_1, \varepsilon) \in L$ acting by

$$(A_1, g_1, \varepsilon)(X,Y) = (A_1 \sigma^\varepsilon X, g_1 \sigma^\varepsilon Y).$$

The group $Spin(1) \cong O(1)$ is embedded in $Spin(2)$ as $\{\pm 1\} \subset S^1$. An element of $M$ is then given by $(A_0, g_0, \varepsilon)$ with $A_0 \in U(n-1)$, $g_0 \in O(1) = \{\pm 1\}$ and $\varepsilon \in \mathbf{Z}_2$, where

$$(A_0, g_0, \varepsilon)(X,Y) = (g_0 A_0 \sigma^\varepsilon X, g_0^2 \sigma^\varepsilon Y) = (\pm A_0 \sigma^\varepsilon X, \sigma^\varepsilon Y)$$

for all $(X,Y) \in \mathfrak{v} \times (\mathfrak{z} \oplus \mathfrak{a})$. Replacing $\pm A_0 \in U(n-1)$ by $A_1$, we see that $M \cong U(n-1) \times \mathbf{Z}_2$ with $(A_1, \varepsilon) \in M$ acting by

$$(A_1, \varepsilon)(X,Y) = (A_1 \sigma^\varepsilon X, \sigma^\varepsilon Y).$$

To describe the action of the pair $(\lambda, W)$ with $\lambda \in \mathbf{R}$, $W \in \mathbf{C}^{n-1}$ and $\lambda^2 + |W|^2 = 1$ on $(X,Y) \in \mathfrak{v} \times (\mathfrak{z} \oplus \mathfrak{a})$, we first write $X = \mu W + W'$ where $\mu \in \mathbf{C}$ and $(W, W')_{\mathbf{C}} = 0$. (Explicitly, $\mu = (X,W)_{\mathbf{C}} / |W|^2$ and $W' = X - \mu W$.) Then

$$(\lambda, W)(X,Y) = (YW - \lambda \mu W + W', \lambda Y + \mu).$$

By composing all such pairs $(\lambda, W)$ with all $l \in L$, it is clear that the group $K$ is isomorphic to $U(n) \times \mathbf{Z}_2$ with $(A_0, \varepsilon) \in U(n) \times \mathbf{Z}_2$ acting on $V \in \mathbf{C}^n \cong \mathfrak{v} \oplus (\mathfrak{z} \oplus \mathfrak{a})$ by

$$(A_0, \varepsilon)V = A_0 \sigma^\varepsilon V.$$

The action of $a_u \in A$ on $B = \{p \in \mathfrak{s} : |p| < 1\}$ is given by

$$a_u(X,Y) = (X(sY+c)^{-1}, (cY+s)(sY+c)^{-1})$$

for all $(X,Y) \in B \subset \mathbf{C}^{n-1} \times \mathbf{C}$, where $c = (u+1)/(2\sqrt{u})$, $s = (u-1)/(2\sqrt{u})$ as in Section 3.1.2. By the Cartan decomposition, we see that the full group $G$ of isometries of $B$ is indeed isomorphic to $PU(1,n) \times \mathbf{Z}_2$ where

$$\left( \begin{pmatrix} a & b^* \\ c & D \end{pmatrix}, \varepsilon \right) p = (c + D\sigma^\varepsilon p)(a + b^* \sigma^\varepsilon p)^{-1}$$

for all $p \in \mathbf{C}^n \cong \mathfrak{s}$ and $a \in \mathbf{C}$, $b, c \in \mathbf{C}^n$, $D \in \mathbf{C}(n)$ such that the matrix is an element of $U(1,n)$ (clearly multiplication of such a matrix by a unit complex number does not affect its action on $\mathfrak{s}$).

### 3.3.3 Case 3: $Sp(1,n)/Sp(n)$

Suppose $q = \dim(\mathfrak{z}) = 3$. The group $Spin(4) \cong Sp(1) \times Sp(1)$ acts orthogonally on $Q' \cong \mathbf{H}$ (where the isomorphism is a vector space isomorphism) by

$$\pi_O \begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix} x = qx\overline{r}$$

for all $q, r \in S^3$, $x \in \mathbf{H}$. (Recall that $\widehat{x} = jxj^{-1}$ and $\tilde{x} = \widehat{\overline{x}}$ for all $x \in \mathbf{H}$.) One of the two Clifford actions on $\mathfrak{v} = \mathbf{H}^{n-1}$ is given by

$$\pi_C \begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix} X = X\overline{r}$$

for all $q, r \in S^3$, $X \in \mathbf{H}^{n-1}$. Identify $\mathfrak{z}$ with $\mathrm{Im}(\mathbf{H}) \cong \mathbf{R}^3$ and $\mathfrak{a}$ with $\mathrm{Re}(\mathbf{H}) \cong \mathbf{R}$. The sphere $S^3$ is contained in $Spin(4)$ by

$$q \mapsto \begin{pmatrix} q & 0 \\ 0 & \tilde{q} \end{pmatrix}.$$

It follows that $J_Z X = XZ$ for all $X \in \mathfrak{v} = \mathbf{H}^{n-1}$, $Z \in \mathfrak{z} = \mathbf{R}^3$. The $J^2$ condition holds by the associativity of $\mathbf{H}$. Given $X, Y \in \mathfrak{v}$, we must have $[X, Y] = Z$ for some $Z \in \mathbf{R}^3$. For any $W \in \mathfrak{z}$,

$$\begin{aligned} \mathrm{Re}(\overline{W}Z) &= \langle W, Z\rangle = \langle W, [X, Y]\rangle = \langle XW, Y\rangle \\ &= \mathrm{Re}\left(\overline{W}(X, Y)_{\mathbf{H}}\right) = \mathrm{Re}\left(\overline{W}(\mathrm{Im}(X, Y)_{\mathbf{H}})\right) \end{aligned}$$

where

$$(X, Y)_{\mathbf{H}} = \sum_{j=1}^{n-1} \overline{X}_j Y_j, \qquad \langle X, Y\rangle = \mathrm{Re}(X, Y)_{\mathbf{H}}, \qquad \langle Z_1, Z_2\rangle = \mathrm{Re}(\overline{Z}_1 Z_2)$$

for all $X, Y \in \mathfrak{v}$, $Z_1, Z_2 \in \mathfrak{z} \oplus \mathfrak{a}$. It follows that

$$[X, Y] = \mathrm{Im}(X, Y)_{\mathbf{H}}$$

for all $X, Y \in \mathfrak{v}$. For any $p = (X, Z + t) \in \mathbf{H}^n$, with $Z \in \mathbf{R}^3$ and $t \in \mathbf{R}$, we have

$$T_p^{(2)} \oplus \mathbf{R}p = \left\{\left(X(t - Z)(u + W), \left(|Z|^2 + t^2\right)(W + u)\right) : W \in \mathbf{R}^3, u \in \mathbf{R}\right\} = p\mathbf{H}.$$

We seek orthogonal automorphisms of $\mathfrak{v}$ which commute or anticommute with $J_Z$, for all $Z \in \mathbf{R}^3$. In the former case, we see that the automorphisms are precisely the elements of $Sp(n-1)$. We claim that there are no automorphisms which anticommute with the Clifford action, that is, there does not exist an orthogonal map $g : \mathbf{H}^n \to \mathbf{H}^n$ such that

$$g(X)Z = -g(XZ)$$

for all $X \in \mathbf{H}^n$, $Z \in \mathrm{Im}(\mathbf{H})$. Suppose such a $g$ exists. Set $X = e = (1, 0, \dots, 0)$. Then $g(Ze) = -g(e)Z$ for all $Z \in \mathrm{Im}(\mathbf{H})$, so by linearity, $g(Ye) = g(e)\overline{Y}$ for all $Y \in \mathbf{H}$. This gives

$$g(e)\overline{Y}Z = -g(e)\overline{YZ}$$

for all $Y \in \mathbf{H}$, $Z \in \mathrm{Im}(\mathbf{H})$. Cancelling $g(e) \neq 0$ and using linearity again, we have

$$\overline{Y}Z = Z\overline{Y}$$

for all $Y, Z \in \mathbf{H}$ which is a contradiction.

We have now established that any element of $L$ is given by $\left(\begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix}, A\right)$ with $\begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix} \in Spin(4), A \in Sp(n-1)$, where

$$\begin{pmatrix} q & 0 \\ 0 & \widehat{r} \end{pmatrix}(X, Y) = (X\overline{r}, qY\overline{r})$$

for all $(X, Y) \in \mathfrak{v} \times (\mathfrak{z} \oplus \mathfrak{a})$. As a result, we can write $L \cong Sp(n-1) \times Sp(1) \times Sp(1)$ with $(A, q, r) \in L$ acting by

$$(A, q, r)(X, Y) = (AX\overline{r}, qY\overline{r}).$$

The group $Spin(3) \cong Sp(1)$ is embedded in $Spin(4)$ by

$$q \mapsto \begin{pmatrix} q & 0 \\ 0 & \widehat{q} \end{pmatrix}.$$

Any element of $M$ is then given by $(q, A)$ with $q \in Sp(1)$, $A \in Sp(n-1)$, where

$$(q, A)(X, Y) = (AX\overline{q}, qY\overline{q})$$

for all $(X, Y) \in \mathfrak{v} \times (\mathfrak{z} \oplus \mathfrak{a})$. That is, $M \cong Sp(n-1) \times Sp(1)$. In particular, if $q = Z \in \mathbf{R}^3 \cap Sp(1)$ and $A = -I$, then $\overline{Z} = Z^{-1} = -Z$ and

$$(q, A)(X, Y) = (XZ, ZYZ^{-1}) = (J_Z X, -\rho_Z Y)$$

for all $(X, Y) \in \mathfrak{v} \times (\mathfrak{z} \oplus \mathfrak{a})$.

To describe the action of the pair $(\lambda, W)$, where we have $\lambda \in \mathbf{R}, W \in \mathbf{H}^{n-1}$ and $\lambda^2 + |W|^2 = 1$ on $(X, Y) \in \mathfrak{v} \times (\mathfrak{z} \oplus \mathfrak{a})$, we first write $X = W\mu + W'$ where $\mu \in \mathbf{H}$ and $(W, W')_{\mathbf{H}} = 0$. (Explicitly, $\mu = (W, X)_{\mathbf{H}} / |W|^2$ and $W' = X - W\mu$.) Then

$$(\lambda, W)(X, Y) = (WY - \lambda W\mu + W', \lambda Y + \mu).$$

By composing all such pairs $(\lambda, W)$ with all $l \in L$, it is clear that the group $K$ is isomorphic to $Sp(n) \times Sp(1)$, where $(A, q) \in Sp(n) \times Sp(1)$ acts on $\mathfrak{v} \oplus (\mathfrak{z} \oplus \mathfrak{a}) \cong \mathbf{H}^n$ by

$$X \mapsto Ax\overline{q}$$

for all $X \in \mathbf{H}^n$. Note that due to the noncommutativity of $\mathbf{H}$, there does not in general exist $A_0 \in Sp(n)$ such that $Ax\overline{q} = A_0 x$ for all $x$.

The action of $a_u \in A$ on the ball $B$ is given by

$$(X, Y) \mapsto (X(sY + c)^{-1}, (cY + s)(sY + c)^{-1})$$

for all $(X, Y) \in B \subset \mathbf{H}^{n-1} \times \mathbf{H}$, where $c = (u+1)/(2\sqrt{u})$, $s = (u-1)/(2\sqrt{u})$ as before. Using the Cartan decomposition, we see that the full group of isometries of $B$ is $Sp(1, n) \times Sp(1)$ acting on $B$ by

$$\left( \begin{pmatrix} a & b^* \\ c & D \end{pmatrix}, q \right) p = (c + Dp)(a + b^* p)^{-1} \overline{q}$$

for all $p \in \mathfrak{s} \cong \mathbf{H}^n$, $q \in Sp(1)$ and $a \in \mathbf{H}, b, c \in \mathbf{H}^n, D \in \mathbf{H}(n)$ such that the matrix is an element of $Sp(1, n)$.

### 3.3.4 Case 4: $F_{4(-20)}/Spin(9)$

Suppose $q = \dim(\mathfrak{z}) = 7$. Identify $\mathfrak{z} \oplus \mathfrak{a}$ with $\mathbf{Y}$ by setting $H = I$ (the 8 by 8 identity matrix) and $\mathfrak{z} = \nu(\mathbf{R}^7)$. The group $Spin(8) \subset SO(8) \times SO(8)$ acts orthogonally on $\mathbf{Y}$ by

$$\pi_O \begin{pmatrix} g_0 & 0 \\ 0 & \breve{g}_1 \end{pmatrix} y = g_0 y \breve{g}_1^{-1}$$

for all $(g_0, g_1, g_2) \in Spin(8)$, $x \in \mathbf{O} \cong \mathbf{R}^8$. One of the two Clifford actions on $\mathfrak{v} = \mathbf{O}$ is given by

$$\pi_C \begin{pmatrix} g_0 & 0 \\ 0 & \breve{g}_1 \end{pmatrix} X = g_1 X$$

for all $(g_0, g_1, g_2) \in Spin(8)$, $X \in \mathbf{O}$. The sphere $S^7$ is contained in $Spin(8)$ by

$$X \mapsto \begin{pmatrix} \nu(X) & 0 \\ 0 & \nu(X)^t \end{pmatrix}.$$

Then $J_Z X = (\nu(Z)^t)\check{\ } X = \nu(X)Ze = X(Ze)$ for all $X \in \mathfrak{v} = \mathbf{O}$, $Z \in \mathfrak{z} = \nu(\mathbf{R}^7)$. The $J^2$ condition holds since $(t + J_Z)\mathbf{O} = \mathbf{O}$ for all $t \in \mathbf{R}, Z \in \mathfrak{z}$, although as we shall see this would not be the case if $\mathfrak{v}$ were the (reducible) Clifford module $\mathbf{O}^n$ for any $n > 1$. Given $X, Y \in \mathfrak{v}$, let $[X, Y] = Z \in \nu(\mathbf{R}^7)$. For any $W \in \mathfrak{z}$,

$$\begin{aligned} \mathrm{Re}(\overline{W}Ze) &= \langle W, Z\rangle\, e = \langle W, [X,Y]\rangle\, e = \langle \nu(X)We, Y\rangle\, e \\ &= \mathrm{Re}(\overline{X(We)}Y) = \mathrm{Re}(\overline{W}\,\overline{X}Y) = \mathrm{Re}(\overline{W}\mathrm{Im}(\overline{X}Y)). \end{aligned}$$

It follows that

$$[X, Y] = \mathrm{Im}\left(\overline{X}Y\right) = \mathrm{Im}\left(\nu(X)^t Y\right)$$

for all $X, Y \in \mathfrak{v} \cong \mathbf{O}$. For any $p = (X, \nu(Y)) \in \mathfrak{v} \oplus (\mathfrak{z} \oplus \mathfrak{a})$, with $X \in \mathfrak{v}$ and $Y \in \mathbf{O}$, we have

$$T_p^{(2)} \oplus \mathbf{R}p = \{((XY^{-1})W, \nu(W)) : W \in \mathbf{O}\}.$$

Note that $T_p^{(2)} \oplus \mathbf{R}p$ is not in general equal to $p\mathbf{O}$ since

$$(XY^{-1})W \neq X(Y^{-1}W)$$

by the nonassociativity of $\mathbf{O}$. This explains why the $J^2$ condition does not hold for $\dim_{\mathbf{O}} \mathfrak{v}$ greater than 1.

We claim that there are no orthogonal intertwining operators of $\mathfrak{v}$ apart from $\pm I$. Clearly any such operator satisfies

$$g(X)Z = g(XZ)$$

for all $X \in \mathbf{O}, Z \in \mathrm{Im}(\mathbf{O})$. Setting $X = 1$, we have $g(Z) = aZ$ where $a = g(1) \in \mathbf{O}$ with $|a| = 1$. This also holds for $Z \in \mathbf{R}$. By linearity, it follows that $(aX)Z = a(XZ)$ for all $X, Z \in \mathbf{O}$. By Lemma 2.10(iv), $a \in \mathbf{R}$, whence $g(X) = \pm X$ for all $X \in \mathfrak{v}$ as claimed.

We now demonstrate that, as in the case of $Sp(1, n)/Sp(n)$, there are no orthogonal automorphisms of $\mathfrak{v}$ which anticommute with $J_Z$ for all $Z \in \mathfrak{z}$. Clearly such an operator satisfies

$$g(X)Z = -g(XZ)$$

for all $X \in \mathbf{O}$, $Z \in \mathrm{Im}(\mathbf{O})$. Setting $X = 1$, we have $g(Z) = -aZ$ where $a = g(1) \in \mathbf{O}$ with $|a| = 1$. By linearity, $g(X) = a\overline{X}$, so we have

$$(a\overline{X})\, Z = a\,(Z\overline{X})$$

for all $X, Z \in \mathbf{O}$. If $X = \overline{a}$, then by alternativity we have

$$a(aZ) = (aa)Z = a(Za)$$

whence $aZ = Za$ for all $Z \in \mathbf{O}$. This implies that $a \in \mathbf{R}$, so

$$\overline{X}Z = Z\overline{X}$$

for all $X, Z \in \mathbf{O}$ which is a contradiction.

In light of Theorem 3.2, we have now established that $L \cong \mathit{Spin}(8)$ where the $\theta$-triad $(g_0, g_1, g_2)$ acts on $\mathfrak{v} \oplus (\mathfrak{z} \oplus \mathfrak{a})$ by

$$(X, Y) \mapsto (g_1 X, g_0 Y \breve{g}_1^{-1}).$$

Recall that $\mathit{Spin}(7)$ may be embedded in $\mathit{Spin}(8)$ by

$$\mathit{Spin}(7) \cong H_2 = \{(g_0, g_1, g_2) \in \mathit{Spin}(8) : g_2 e = e\}.$$

Any $\theta$-triad in $H_2$ satisfies

$$g_0 \nu(e) \breve{g}_1^{-1} e = g_2 \nu(e) e = \overline{g_2 \overline{e}} = \overline{g_2 e} = \overline{e} = e,$$

whence

$$(g_0, g_1, g_2)(0, \nu(e)) = (0, g_0 I \breve{g}_1^{-1}) = (0, \nu(e)).$$

This shows that $H_2$ fixes $H$. Since $M \cong \mathit{Spin}(7)$ by Theorem 3.1, we have $M = H_2$. In particular, for any $Z \in \mathfrak{z}$ with $|Ze| = 1$, the $\theta$-triad $(-Z, (Z^t)\breve{}\,, -Z^t \breve{Z})$ is in $H_2$ since

$$-Z^t \breve{Z} e = -Z^t \overline{Z} e = Z^t Z e = |Z|^2\, e = e.$$

The action of this $\theta$-triad on $(X, Y) \in \mathfrak{v} \oplus (\mathfrak{z} \oplus \mathfrak{a})$ is given by

$$(X, Y) = ((Z^t)\breve{}\, X, -ZY(Z^t)^{-1}) = (XZe, ZYZ^{-1}) = (J_Z X, -\rho_Z Y).$$

(See Lemmas 2.12 and 2.10.)

The pair $(\lambda, W)$ with $\lambda \in \mathbf{R}$, $W \in \mathfrak{v} = \mathbf{O}$, $\lambda^2 + |W|^2 = 1$ acts on $\mathfrak{v} \oplus (\mathfrak{z} \oplus \mathfrak{a})$ by

$$(\lambda, W)(X, Y) = (WYe - \lambda X, \lambda Y + \nu(WX)).$$

Viewing the set of such pairs as a copy of $S^8$, we see that, in light of Lemma 2.5, $K \cong Spin(9)$.

The action of $a_u \in A$ on $B = \{p \in \mathfrak{s} : |p| < 1\}$ is given by

$$(X, Y) \mapsto (X(sY + c)^{-1}, (cY + s)(sY + c)^{-1})$$

for all $(X, Y) \in B \subset \mathbf{C}^{n-1} \times \mathbf{C}$, where $c = (u + 1)/(2\sqrt{u})$, $s = (u - 1)/(2\sqrt{u})$ as before. The full group $G = KAK \cong F_{4(-20)}$ of isometries of $B$ is difficult to describe; indeed, it is hoped that the $H$-type formulation will lead to an easier way of realising this group. Nevertheless we provide the formal definition here, as presented in Takahashi [T]. We seek to find the octonionic equivalent of $O(1, n)$, $U(1, n)$ and $Sp(1, n)$ for the case when $n = 2$. Accordingly, let $J_{1,2}$ denote the (Jordan) algebra of $3 \times 3$ hermitian matrices with coefficients in $\mathbf{O} \otimes \mathbf{C}$ of the form

$$\begin{pmatrix} a_1 & u_3 \otimes i & \overline{u}_2 \otimes i \\ \overline{u}_3 \otimes i & a_2 & u_1 \\ u_2 \otimes i & \overline{u}_1 & a_3 \end{pmatrix}, \quad a_i \in \mathbf{R}, \quad u_i \in \mathbf{O}, \quad i = 1, 2, 3,$$

where conjugation is given by $\overline{(a \otimes \alpha)} = \overline{a} \otimes \alpha$ for all $a \otimes \alpha \in \mathbf{O} \otimes \mathbf{C}$ and the product is given by $x \circ y = \frac{1}{2}(xy + yx)$. The group $F_{4(-20)}$ is then defined to be the connected component of the identity of the group of automorphisms of $J_{1,2}$. It acts on the unit ball in $\mathbf{O}^2$ by restriction when this ball is embedded in $J_{1,2}$ by

$$(x, y) \mapsto \lambda^{-2} \begin{pmatrix} 1 & \overline{y} \otimes i & \overline{x} \otimes i \\ y \otimes i & -|y|^2 & -y\overline{x} \\ x \otimes i & -x\overline{y} & -|x|^2 \end{pmatrix}$$

where $\lambda^2 = 1 - |x|^2 - |y|^2$.

## 3.4 Graded Automorphisms of $\mathfrak{n}$

Pansu [Pu] has proved that every quasiconformal map of the boundary of one of the symmetric spaces $Sp(1, n)/Sp(n)$ or $F_{4(-20)}/Spin(9)$ is conformal with respect to the boundary metric which is given in Section 4.2.1. (The definition of quasiconformality is given in Section 4.2.2.) Pansu's result involves a lemma (Proposition 10.1 on

p. 41 of [Pu]) involving graded automorphisms of the associated space $\mathfrak{n}$. We have already seen in Theorem 3.1 that such a graded automorphism is an element of the group $M$ provided that it is orthogonal. Pansu's result implies that the orthogonality condition may be relaxed somewhat in the cases when $\dim \mathfrak{z} = 3$ or 7. In particular, for any graded automorphism of $\mathfrak{n}$, there exists a dilation of $\mathfrak{n}$ such that the composition of the dilation and the automorphism is an element of $M$. Pansu's proof of his result relies on particular properties of the Lie algebras of $Sp(n-1)$ and $Spin(7)$ and as such may be thought of as using a case-by-case analysis. We present a new proof, largely based on a result of Saal [S], which requires only that the $J^2$ condition hold and that $\dim \mathfrak{z} \equiv 3 \pmod 4$.

**Lemma 3.4** *For any unit vector $Z' \in \mathfrak{z}$, the map*

$$\varphi_{Z'} : \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a} \to \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}; \quad (X, Z, t) \mapsto (J_{Z'}X, -\rho_{Z'}Z, t)$$

*satisfies $\varphi_{Z'} \in M$.*

**Proof** This is obvious from our construction in Section 3.2, however we may prove the result directly as follows. Since $\varphi_{Z'}$ is orthogonal, we need only check that

$$\varphi_{Z'}(T_p^{(2)} \oplus \mathbf{R}p) = T^{(2)}_{\varphi_{Z'}p} \oplus \mathbf{R}\varphi_{Z'}p$$

for all $p = (X, Z, t) \in \mathfrak{s}$. By (3.21),

$$J_{\rho_{Z'}W} J_{Z'} = -J_{Z'} J_W$$

for all $W \in \mathfrak{z}$. If $v \in T_p^{(2)} \oplus \mathbf{R}p$, then

$$v = \left((u + J_W)(t - J_Z)X, \left(|Z|^2 + t^2\right) W, \left(|Z|^2 + t^2\right) u\right)$$

for some $u \in \mathbf{R}$, $W \in \mathfrak{z}$, so

$$\begin{aligned}
\varphi_{Z'}v &= (J_{Z'}(u + J_W)(t - J_Z)X, -(|Z|^2 + t^2)\rho_{Z'}W, (|Z|^2 + t^2)u) \\
&= ((u + J_{W'})(t - J_{-\rho_{Z'}Z})J_{Z'}X, (|-\rho_{Z'}Z|^2 + t^2)W', (|-\rho_{Z'}Z|^2 + t^2)u) \\
&\in T^{(2)}_{\varphi_{Z'}p} \oplus \mathbf{R}\varphi_{Z'}p,
\end{aligned}$$

where $W' = -\rho_{Z'}W$. Note that we have used the orthogonality of $-\rho_{Z'}$. □

**Theorem 3.5** *If $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is a nondegenerate H-type algebra satisfying the $J^2$ condition with* $\dim(\mathfrak{z}) \equiv 3 \pmod 4$, *then* any *graded automorphism of $\mathfrak{n}$ is the product of a dilation with the restriction to $\mathfrak{n}$ of an element of $M$. That is, if $A \in GL(\mathfrak{v})$ and $B \in GL(\mathfrak{z})$ satisfy*

$$[AX, AY] = B[X, Y]$$

*for all $X, Y \in \mathfrak{n}$, then if $F(X, Z) = (AX, BZ)$ for all $(X, Z) \in \mathfrak{v} \oplus \mathfrak{z}$ we have*

$$F = \delta_\lambda m\big|_{\mathfrak{n}}$$

*for some $\lambda > 0$, $m \in M$, where*

$$\delta_\lambda(X, Z) = (\lambda^{1/2} X, \lambda Z)$$

*for all $(X, Z) \in \mathfrak{v} \oplus \mathfrak{z}$.*

**Proof** Suppose $B = I$, that is,

$$[AX, AY] = [X, Y]$$

for all $X, Y \in \mathfrak{v}$. We claim that $A$ intertwines the representation $J$ of $C^0(q)$ on $\mathfrak{v}$, where $q = \dim(\mathfrak{z})$. Let $\{Z_1, \dots, Z_q\}$ denote an orthonormal basis for $\mathfrak{z}$, and let $J_i = J_{Z_i}$ for $i = 1, \dots, q$. For any $1 \le i, k \le q$, $X, Y \in \mathfrak{v}$ we have

$$\begin{aligned} \langle Z_i, [X, Y]\rangle &= \langle J_i X, Y\rangle = \langle J_k J_i J_k X, Y\rangle \\ &= \langle Z_k, [J_i J_k X, Y]\rangle = \langle Z_k, [A J_i J_k X, AY]\rangle . \end{aligned} \tag{3.24}$$

Replacing $X, Y$ by $AX, AY$ respectively in the first and fourth expressions of (3.24), we have

$$\langle Z_i, [X, Y]\rangle = \langle Z_i, [AX, AY]\rangle = \langle Z_k, [J_i J_k AX, AY]\rangle .$$

By the surjectivity of $A$ it follows that

$$\langle J_k(J_i J_k A - A J_k J_i)X, Y\rangle = \langle Z_k, [(J_i J_k A - A J_k J_i)X, Y]\rangle = 0$$

for all $X, Y \in \mathfrak{v}$. By the surjectivity of the Lie bracket (on $\mathfrak{z}$),

$$A J_i J_k = J_i J_k A$$

for all $1 \le i, k \le q$ as claimed.

By the $J^2$ condition, we may write

$$\mathfrak{v} = \bigoplus_{i=1}^{m} \mathfrak{v}_i$$

where $\mathfrak{v}_i = \mathbf{R}X_i \oplus \mathfrak{j}(X_i)$ for $i = 1, \dots, m$, $\{X_i\}$ being a set of orthonormal vectors in $\mathfrak{v}$, such that $\{\mathfrak{v}_i\}$ is a set of equivalent irreducible Clifford submodules. Let

$$K = J_1 J_2 \cdots J_q.$$

Since $q \equiv 3 \pmod 4$, we see that $K^2 = I$ and $KJ_i = J_iK$ for all $i = 1, \dots, q$. Let $K_i$ denote the restriction of $K$ to $\mathfrak{v}_i$. Since $K_i$ is orthogonal and $K_i^2 = I$, $K_i$ must have an eigenvector $v_i$ with eigenvalue $\varepsilon_i = \pm 1$. By Schur's Lemma, $K_i - \varepsilon_i I = 0$, that is, $K = \pm I$ on each $\mathfrak{v}_i$. By the equivalence of the submodules $\{\mathfrak{v}_i\}$, we see that $K = \pm I$ on $\mathfrak{v}$, thus $J_1 = \pm J_2 \cdots J_q$. Since $AJ_iJ_k = J_iJ_kA$ for all $1 \le i, k \le q$, we have $AJ_1 = J_1A$, thus

$$\langle AJ_1X, AY\rangle = \langle J_1AX, AY\rangle = \langle Z_1, [AX, AY]\rangle = \langle Z_1, [X, Y]\rangle = \langle J_1X, Y\rangle$$

for all $X, Y \in \mathfrak{v}$; the surjectivity of $J_1$ implies that $A$ is orthogonal. Now

$$\langle J_ZX, Y\rangle = \langle Z, [X, Y]\rangle = \langle Z, [AX, AY]\rangle = \langle J_ZAX, AY\rangle = \left\langle A^{-1}J_ZAX, Y\right\rangle$$

for all $X, Y \in \mathfrak{v}$, $Z \in \mathfrak{z}$, whence $AJ_Z = J_ZA$ for all $Z \in \mathfrak{z}$. We claim that the map

$$\varphi : \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a} \to \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}; \quad (X, Z, t) \mapsto (AX, Z, t)$$

satisfies $\varphi \in M$. Since $\varphi$ is orthogonal, this amounts to showing that

$$\varphi\left(T_p^{(2)} \oplus \mathbf{R}p\right) = T_{\varphi p}^{(2)} \oplus \mathbf{R}\varphi p$$

for all $p = (X, Z, t) \in \mathfrak{s}$. If $v \in T_p^{(2)} \oplus \mathbf{R}p$, then

$$v = \left((u + J_W)(t - J_Z)X, \left(|Z|^2 + t^2\right)W, \left(|Z|^2 + t^2\right)u\right)$$

for some $u \in \mathbf{R}$, $W \in \mathfrak{z}$, so

$$\begin{aligned}
\varphi v &= \left(A(u + J_W)(t - J_Z)X, \left(|Z|^2 + t^2\right)W, \left(|Z|^2 + t^2\right)u\right) \\
&= \left((u + J_W)(t - J_Z)AX, \left(|Z|^2 + t^2\right)W, \left(|Z|^2 + t^2\right)u\right) \\
&\in T_{\varphi p}^{(2)} \oplus \mathbf{R}\varphi p.
\end{aligned}$$

The result is therefore true in this case with $\lambda = 1$.

Now suppose that $A, B$ satisfy the hypotheses of the theorem and that $\det(B)$ is positive. Write

$$B = T_1 B_0 T_2$$

where $T_1, T_2$ are orthogonal and $B_0$ is diagonal with positive eigenvalues $\lambda_1, \ldots, \lambda_q$. Since $\det(B) > 0$, we can ensure that $\det(T_1) = \det(T_2) = 1$. We may then write $T_1$ and $T_2$ as the product of an even number of reflections in hyperplanes perpendicular to given elements of $\mathfrak{z}$, that is,

$$T_1 = (-\rho_{Z_{i_1}}) \cdots (-\rho_{Z_{i_r}})$$

for some unit vectors $Z_{i_1}, \ldots, Z_{i_r} \in \mathfrak{z}$, and similarly for $T_2$, where $\rho_Z$ is (as usual) the reflection in the hyperplane $(\mathbf{R}Z)^\perp$. The identity (3.10)

$$[J_Z X, J_Z Y] = -\rho_Z [X, Y]$$

for all $X, Y \in \mathfrak{v}$, $Z \in \mathfrak{z}$, $|Z| = 1$ implies that $(J_Z, -\rho_Z)$ is a graded automorphism of $\mathfrak{v} \oplus \mathfrak{z}$. Furthermore $(J_Z, -\rho_Z)$ is the restriction to $\mathfrak{n}$ of the map $(J_Z, -\rho_Z, I)$ which is in $M$ by Lemma 3.4. Consequently

$$(A, B) = m_1|_{\mathfrak{n}} \, (A_0, B_0) \, m_2|_{\mathfrak{n}} \tag{3.25}$$

for some $m_1, m_2 \in M$ and $A_0 \in GL(\mathfrak{v})$, where $(A_0, B_0)$ is a graded automorphism of $\mathfrak{v} \oplus \mathfrak{z}$. Now

$$\begin{aligned} \left\langle A_0^t J_i A_0 X, Y \right\rangle &= \langle J_i A_0 X, A_0 Y \rangle = \langle Z_i, [A_0 X, A_0 Y] \rangle = \langle Z_i, B_0 [X, Y] \rangle \\ &= \langle B_0 Z_i, [X, Y] \rangle = \lambda_i \langle Z_i, [X, Y] \rangle = \lambda_i \langle J_i X, Y \rangle \end{aligned}$$

for all $X, Y \in \mathfrak{v}$, $1 \le i \le q$, so $A_0^t J_i A_0 = \lambda_i J_i$ for all such $i$. It follows that $\lambda_i^n = (\det(A_0))^2$ for all $1 \le i \le q$, where $n = \dim(\mathfrak{v})$. That is, $B_0 = \lambda I$ for some $\lambda > 0$, whence

$$(A, B) = \delta_\lambda \, m_1|_{\mathfrak{n}} \, (A_1, I) \, m_2|_{\mathfrak{n}} \, ,$$

where $(A_1, I)$ is a graded automorphism of $\mathfrak{n}$. The above special case demonstrated that $(A_1, I) = m|_{\mathfrak{n}}$ for some $m \in M$, thus the result follows in this case.

Finally, suppose $\det(B) < 0$. Since $q = \dim(\mathfrak{z})$ is odd, we may write

$$B = T_1 B_0 T_2$$

where $T_1, T_2 \in SO(\mathfrak{z})$ and $B_0$ is diagonal with negative eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q < 0$. As before,

$$(A, B) = m_1|_{\mathfrak{n}} (A_0, B_0)\, m_2|_{\mathfrak{n}}$$

for some $m_1, m_2 \in M$ and $A_0 \in GL(\mathfrak{v})$, where $(A_0, B_0)$ is a graded automorphism of $\mathfrak{v} \oplus \mathfrak{z}$. By the argument following (3.25), we see that $\lambda_i^n = (\det(A_0))^2$ for all $1 \leq i \leq q$, so that $B_0 = \lambda I$ for some $\lambda < 0$, whence there exists $A_1 \in GL(\mathfrak{v})$ such that $(A_1, -I)$ is a graded automorphism of $\mathfrak{v} \oplus \mathfrak{z}$. It follows that $A_1^t J_Z A_1 = -J_Z$ for all $Z \in \mathfrak{z}$. By a similar argument to one used at the beginning of this proof, we can show that

$$A_1 J_i J_k = J_i J_k A_1$$

for all $1 \leq i, k \leq q$. It follows that

$$A_1^t K A_1 = -K$$

where $K = J_1 \cdots J_q$ as before. We have already demonstrated that $K = \pm I$. It follows that

$$A_1^t A_1 = -I$$

which is clearly impossible, for if $X \in \mathfrak{v} \backslash \{0\}$ then

$$-\langle X, X \rangle = \langle A_1^t A_1 X, X \rangle = \langle A_1 X, A_1 X \rangle \geq 0$$

which is a contradiction. It follows that $\det(B) > 0$, establishing the result. $\square$

In light of this theorem and Theorem 3.1(i), we have the following result.

**Corollary** *Let* $\mathrm{Aut}_{\mathfrak{v}}(\mathfrak{n})$ *denote the set of graded automorphisms of* $\mathfrak{n}$ *where* $\mathfrak{n}$ *is as in Theorem 3.5. Then*

$$M \cong \mathrm{Aut}_{\mathfrak{v}}(\mathfrak{n}) / \mathbf{R}^+.$$

Essentially, Pansu's argument shows that any quasiconformal map of the boundary of $Sp(1,n)/Sp(n)$ or $F_{4(-20)}/Spin(9)$ has a derivative almost everywhere which, where it exists, is a graded automorphism of the tangent space, considered as a Lie algebra isomorphic to $\mathfrak{n}$. The decomposition of this derivative as the product of a dilation and an isometry provides the requisite conformality. See [Pu] for details.

# Chapter 4

# The Geometry of the Spaces Constructed

In this chapter we examine the Riemannian geometry of the symmetric spaces constructed, finding the geodesics and two formulae for distances in $B$. We also define distance formulae on the boundaries of $B$ and $D$ and establish that the Cayley transform extends to a quasiconformal map with respect to these distances. Except where noted we assume that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is an $H$-type algebra satisfying the $J^2$ condition.

## 4.1 Geodesics and Distance Formulae

In the previous chapter we defined a Riemannian metric on $B$, the unit ball of $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$. In this section we find the associated geodesics and two equivalent formulae for the distances in $B$. One of these distance formulae is not *a priori* symmetric; the other uses the $J^2$ condition to provide the symmetry.

Let $P_{\mathfrak{z}}$ denote the orthogonal projection onto $\mathfrak{z}$. For any $p \in \mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, let $P_p$ denote the orthogonal projection onto $\mathbf{R}p \oplus T_p^{(2)}$. (See (3.18) and (3.19).)

**Definition** For any $p_i = (X_i, Z_i, t_i) \in \mathfrak{s}$, $i = 1, 2, 3$, where $X_3 \neq 0$, we define the *J-product* $\{p_1, p_2\}_{p_3}$ to be the element of $\mathfrak{z}$ such that

$$J_{Z_1} J_{Z_2} X_3 = J_{\{p_1, p_2\}_{p_3}} X_3 - \langle Z_1, Z_2 \rangle X_3. \tag{4.1}$$

Note that $\{p_1, p_2\}_{p_3}$ exists, for if $Z_1 = Z_1' \oplus Z_1''$ with $Z_1' = \langle Z_1, Z_2\rangle |Z_2|^{-2} Z_2$, then

$$J_{Z_1} J_{Z_2} X_3 = J_{Z_1''} J_{Z_2} X_3 + \frac{\langle Z_1, Z_2\rangle}{|Z_2|^2} J_{Z_2}^2 X_3 = J_{Z'''} X_3 - \langle Z_1, Z_2\rangle X_3$$

for some $Z''' \in \mathfrak{z}$ by the $J^2$ condition, noting that $Z_1'' \perp Z_2$ and that $J_{Z_2}^2 = -|Z_2|^2 I$.

We now establish a series of technical lemmas.

**Lemma 4.1** *For any $p_i = (X_i, Z_i, t_i) \in \mathfrak{s}$ for $i = 1, 2$, $\langle P_{p_2} p_1, p_2\rangle = \langle p_1, p_2\rangle$.*

**Proof** By the selfadjointness of the projection operator $P_{p_2}$,

$$\langle P_{p_2} p_1, p_2\rangle = \langle p_1, P_{p_2} p_2\rangle = \langle p_1, p_2\rangle$$

since $p_2 \in \mathbf{R}p_2 \oplus T_{p_2}^{(2)}$. □

**Lemma 4.2** *For all $t \in \mathbf{R}$, $Z \in \mathfrak{z}$, $X, Y \in \mathfrak{v}$, we have*

$$\left(t^2 + |Z|^2\right)\left(\langle X, Y\rangle^2 + |[X,Y]|^2\right) = \langle X, (t + J_z)Y\rangle^2 + |[X, (t+J_z)Y]|^2 .$$

**Proof** The result is trivial if $X = 0$, so we assume otherwise. Express $Y$ in the form $aX + J_Z X + Y'$ for some $a \in \mathbf{R}$, $Z \in \mathfrak{z}$, $Y' \in (\mathbf{R}X \oplus \mathfrak{j}(X))^\perp$. For any $W \in \mathfrak{z}$, we have

$$\langle W, [X, Y']\rangle = \langle J_W X, Y'\rangle = 0,$$

that is, $[X, Y'] = 0$. It follows that

$$[X, Y] = [X, J_Z X] = |X|^2 Z$$

by equation (3.9). This implies that

$$J_{[X,Y]} X = |X|^2 J_Z X = |X|^2 P_{\mathfrak{j}(X)} Y.$$

Since

$$|[X,Y]|^2 = \langle J_{[X,Y]} X, Y\rangle = |X|^2 \langle P_{\mathfrak{j}(X)} Y, Y\rangle ,$$

we see that

$$\langle X, Y\rangle^2 + |[X,Y]|^2 = |X|^2 \langle P_{\mathbf{R}X \oplus \mathfrak{j}(X)} Y, Y\rangle = |Y|^2 \langle P_{\mathbf{R}Y \oplus \mathfrak{j}(Y)} X, X\rangle$$

by symmetry. Replacing $Y$ by $(t + J_Z)Y$, the result follows by the $J^2$ condition. □

**Lemma 4.3** *Let $p_i = (X_i, Z_i, t_i) \in \mathfrak{s}, i = 1, 2$. If $X \in \mathfrak{v}\backslash\{0\}$, then*

$$|\{p_1, p_2\}_X|^2 = |Z_1|^2 |Z_2|^2 - \langle Z_1, Z_2\rangle^2 .$$

**Proof** By definition,

$$J_{Z_1} J_{Z_2} X = J_{\{p_1,p_2\}_X} X - \langle Z_1, Z_2\rangle X,$$

however $\left\langle J_{\{p_1,p_2\}_X} X, X\right\rangle = 0$ and $|J_{Z_1} J_{Z_2} X| = |Z_1| |Z_2| |X|$, from which the result follows easily. □

**Lemma 4.4** *Under the hypotheses of Lemma 4.3, $\langle \{p_1, p_2\}_X, Z_i\rangle = 0$ for $i = 1, 2$.*

**Proof** We have

$$|X|^2 \langle \{p_1, p_2\}_X, Z_2\rangle = \left\langle J_{\{p_1,p_2\}_X} X, J_{Z_2} X\right\rangle = \langle J_{Z_1} J_{Z_2} X + \langle Z_1, Z_2\rangle X, J_{Z_2} X\rangle = 0$$

since $\langle J_{Z'} Y, Y\rangle = 0$ for any $Y \in \mathfrak{v}$, $Z' \in \mathfrak{z}$. Furthermore the identity

$$J_{Z_1} J_{Z_2} + J_{Z_2} J_{Z_1} = -2 \langle Z_1, Z_2\rangle I$$

implies that

$$\begin{aligned} J_{\{p_2,p_1\}_X} X - \langle Z_1, Z_2\rangle X = J_{Z_2} J_{Z_1} X &= -J_{Z_1} J_{Z_2} X - 2 \langle Z_1, Z_2\rangle X \\ &= -J_{\{p_1,p_2\}_X} X - \langle Z_1, Z_2\rangle X, \end{aligned}$$

that is, $\{p_1, p_2\}_X = -\{p_2, p_1\}_X$ which completes the proof (by symmetry). □

**Lemma 4.5** *Under the hypotheses of Lemma 4.3, if $X_1, X_2 \neq 0$, then*

$$\langle \{p_1, p_2\}_{p_i}, [X_1, X_2]\rangle = \langle J_{Z_1} X_1, J_{Z_2} X_2\rangle - \langle Z_1, Z_2\rangle \langle X_1, X_2\rangle$$

*for $i = 1, 2$.*

**Proof** Now

$$\begin{aligned} \langle \{p_1, p_2\}_{p_2}, [X_1, X_2]\rangle &= -\left\langle J_{\{p_1,p_2\}_{p_2}} X_2, X_1\right\rangle \\ &= -\langle J_{Z_1} J_{Z_2} X_2 + \langle Z_1, Z_2\rangle X_2, X_1\rangle \\ &= \langle J_{Z_1} X_1, J_{Z_2} X_2\rangle - \langle Z_1, Z_2\rangle \langle X_1, X_2\rangle . \end{aligned}$$

Since

$$\langle \{p_2, p_1\}_{p_2}, [X_2, X_1]\rangle = \langle -\{p_1, p_2\}_{p_2}, -[X_1, X_2]\rangle ,$$

the result follows by relabelling. □

The following result links the two forms of the distance formula given in Theorem 4.10.

**Theorem 4.6** *For any $p_1, p_2 \in \mathfrak{s}$, $p_2 \neq 0$, $p_i = (X_i, Z_i, t_i)$ for $i = 1, 2$,*

$$|(1 - \langle p_1, p_2\rangle)\, H + P_{\mathfrak{z}}\, [p_1, p_2] + \{p_1, p_2\}_p| = \left| |p_2|\, P_{p_2} p_1 - \frac{p_2}{|p_2|} \right|,$$

*where $p = p_1$ or $p_2$ if $X_1, X_2 \neq 0$, or $p \in \mathfrak{s}\backslash\{0\}$ is arbitrary if either $X_1, X_2 = 0$.*

**Proof** The square of the left-hand expression is

$$\begin{aligned}(1 - \langle p_1, p_2\rangle)^2 &+ |P_{\mathfrak{z}}\, [p_1, p_2] + \{p_1, p_2\}_p|^2 \\ &= 1 - 2\,\langle p_1, p_2\rangle + \langle p_1, p_2\rangle^2 + |P_{\mathfrak{z}}\, [p_1, p_2] + \{p_1, p_2\}_p|^2,\end{aligned}$$

whereas the square of the right-hand expression is

$$|p_2|^2\, |P_{p_2} p_1|^2 - 2 \left\langle |p_2|\, P_{p_2} p_1, \frac{p_2}{|p_2|} \right\rangle + 1 = |p_2|^2\, |P_{p_2} p_1|^2 - 2\,\langle p_1, p_2\rangle + 1$$

by Lemma 4.1. We need only show that

$$|p_2|^2\, |P_{p_2} p_1|^2 = \langle p_1, p_2\rangle^2 + |P_{\mathfrak{z}}\, [p_1, p_2] + \{p_1, p_2\}_p|^2. \tag{4.2}$$

If $(Z_2, t_2) \neq 0$ then an orthonormal basis for $\mathbf{R}p_2 \oplus T_{p_2}^{(2)}$ is given by the vector

$$\{c\left((t_2 - J_{Z_2})X_2, 0, |Z_2|^2 + t_2^2\right)\}$$

and the vectors

$$\{c\left(J_{Y_i}(t_2 - J_{Z_2})X_2, \left(|Z_2|^2 + t_2^2\right) Y_i, 0\right), i = 1, \dots, q\}$$

where $q = \dim(\mathfrak{z})$, $c = \left(|Z_2|^2 + t_2^2\right)^{-1/2} |p_2|^{-1}$ and $\{Y_i\}_{i=1}^q$ is an orthonormal basis for $\mathfrak{z}$. (Note that $|J_{Y_i}(t_2 - J_{Z_2})X_2| = |Y_i|\, |t_2 H - Z_2|\, |X_2| = \left(|Z_2|^2 + t_2^2\right)^{1/2} |X_2|$.) It follows that

$$\begin{aligned}|P_{p_2} p_1|^2 = c^2 \Big[&\left(\langle X_1, (t_2 - J_{Z_2})X_2\rangle + t_1\left(|Z_2|^2 + t_2^2\right)\right)^2 \\ &+ \sum_{i=1}^q \left(\langle X_1, J_{Y_i}(t_2 - J_{Z_2})X_2\rangle + \left(|Z_2|^2 + t_2^2\right)\langle Z_1, Y_i\rangle\right)^2\Big] \\ = c^2 &\left[\langle X_1, (t_2 - J_{Z_2})X_2\rangle^2 + \sum_{i=1}^q \langle Y_i, [X_1, (t_2 - J_{Z_2})X_2]\rangle^2\right. \\ &+ 2t_1\left(|Z_2|^2 + t_2^2\right)\langle X_1, (t_2 - J_{Z_2})X_2\rangle \\ &+ 2\left(|Z_2|^2 + t_2^2\right)\sum_{i=1}^q \langle Z_1, Y_i\rangle\, \langle Y_i, [(t_2 - J_{Z_2})X_2, X_1]\rangle \\ &\left.+ \left(t_1^2 + \sum_{i=1}^q \langle Z_1, Y_i\rangle^2\right)\left(|Z_2|^2 + t_2^2\right)^2\right]\end{aligned}$$

$$
\begin{aligned}
&= c^2 \Big[ \langle X_1, (t_2 - J_{Z_2})X_2\rangle^2 + |[X_1, (t_2 - J_{Z_2})X_2]|^2 \\
&\quad + 2(|Z_2|^2 + t_2^2)\,(t_1 \langle X_1, (t_2 - J_{Z_2})X_2\rangle - \langle Z_1, [X_1, (t_2 - J_{Z_2})X_2]\rangle) \\
&\quad + (t_1^2 + |Z_1|^2)\,(|Z_2|^2 + t_2^2)^2 \Big] \\
&= |p_2|^{-2} \left[ \langle X_1, X_2\rangle^2 + |[X_1, X_2]|^2 + 2\,\langle (t_1 - J_{Z_1})X_1, (t_2 - J_{Z_2})X_2\rangle \right. \\
&\quad \left. + \,(|Z_1|^2 + t_1^2)\,(|Z_2|^2 + t_2^2) \right]
\end{aligned}
$$

by Lemma 4.2. This expression also holds if $(Z_2, t_2) = 0$, for then

$$\mathbf{R}p_2 \oplus T_{p_2}^{(2)} = \mathbf{R}X_2 \oplus \mathfrak{j}(X_2).$$

An orthonormal basis for this subspace is given by

$$\{cX_2\} \cup \{cJ_{Y_i}X_2, i = 1, \dots, q\}$$

where $c = |X_2|^{-1} = |p_2|^{-1}$. Then

$$
\begin{aligned}
|P_{p_2}p_1|^2 &= c^2 \left[ \langle X_1, X_2\rangle^2 + \sum_{i=1}^{q} \langle X_1, J_{Y_i}X_2\rangle^2 \right] \\
&= c^2 \left[ \langle X_1, X_2\rangle^2 + \sum_{i=1}^{q} \langle Y_i, [X_1, X_2]\rangle^2 \right] \\
&= |p_2|^{-2} \left[ \langle X_1, X_2\rangle^2 + |[X_1, X_2]|^2 \right].
\end{aligned}
$$

Since $P_{\mathfrak{z}}[p_1, p_2] = t_1 Z_2 - t_2 Z_1 + [X_1, X_2]$, we have

$$
\begin{aligned}
&\langle p_1, p_2\rangle^2 + |P_{\mathfrak{z}}[p_1, p_2] + \{p_1, p_2\}_p|^2 \\
&\quad = t_1^2 t_2^2 + \langle Z_1, Z_2\rangle^2 + \langle X_1, X_2\rangle^2 + 2t_1t_2\,\langle X_1, X_2\rangle + 2t_1t_2\,\langle Z_1, Z_2\rangle \\
&\qquad + 2\,\langle Z_1, Z_2\rangle\,\langle X_1, X_2\rangle + |\{p_1, p_2\}_p|^2 + t_1^2\,|Z_2|^2 + t_2^2\,|Z_1|^2 \\
&\qquad + |[X_1, X_2]|^2 - 2t_1t_2\,\langle Z_1, Z_2\rangle + 2t_1\,\langle Z_2, [X_1, X_2]\rangle \\
&\qquad - 2t_2\,\langle Z_1, [X_1, X_2]\rangle + 2t_1\,\langle Z_2, \{p_1, p_2\}_p\rangle \\
&\qquad - 2t_2\,\langle Z_1, \{p_1, p_2\}_p\rangle + 2\,\langle [X_1, X_2], \{p_1, p_2\}_p\rangle \\
&\quad = (t_1^2 + |Z_1|^2)\,(t_2^2 + |Z_2|^2) + \langle X_1, X_2\rangle^2 + |[X_1, X_2]|^2 \\
&\qquad + 2\,\langle (t_1 - J_{Z_1})X_1, (t_2 - J_{Z_2})X_2\rangle \\
&\quad = |p_2|^2\,|P_{p_2}p_1|^2
\end{aligned}
$$

by Lemmas 4.3, 4.4 and 4.5, noting that the term $\langle [X_1, X_2], \{p_1, p_2\}_p\rangle = 0$ if either $X_1$ or $X_2$ are zero whereas the other terms involving $\{p_1, p_2\}_p$ are independent of $p$. That is, (4.2) holds and the result is established. □

Before we identify the geodesics of $B$, we develop two more lemmas.

**Lemma 4.7** *Given $X \in \mathfrak{v}\backslash\{0\}$, there exists $k' \in K$ such that $k'(\mathbf{R}X) = \mathfrak{a}$ and $k'(\mathfrak{j}(X)) = \mathfrak{z}$.*

**Proof** By Corollary 6.7 of [CDKR2], if $k' = \exp\left(\frac{\pi}{2}\left(\theta\hat{X} + \hat{X}\right)^{\tilde{}}\right)$ where $\hat{X}$ denotes the unit vector $X/|X|$, then $k'$ is represented by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{ad}\hat{X}^t & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{ad}\hat{X} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the coordinates

$$\mathbf{R}X \oplus \mathbf{R}J_W X \oplus (\mathfrak{j}(X) \ominus \mathbf{R}J_W X) \oplus \mathfrak{k}(X) \oplus \mathbf{R}W \oplus (\mathfrak{z} \ominus \mathbf{R}W) \oplus \mathfrak{a}$$

where $W \in S_{\mathfrak{z}}$ is arbitrary. The lemma follows trivially. □

**Lemma 4.8** *If $t \in \mathbf{R}$, $t' = (1-t)/(1+t)$ and $X \in \mathfrak{v}$, then*

$$\tilde{a}_{t'}(tH + X) = \frac{1}{\sqrt{1-t^2}}X.$$

**Proof** This follows readily from the formula for $\tilde{a}_u$ given in Section 3.1.2, however we may easily calculate it directly. In fact, if $C$ is the Cayley transform given in (3.14), then

$$C(X,0,t) = \left(\frac{2(1-t)}{(1-t)^2}X, 0, \frac{1-t^2}{(1-t)^2}\right) = \left(\frac{2}{1-t}X, 0, \frac{1}{t'}\right),$$

whence

$$a_{t'}C(X,0,t) = \left(\frac{2\sqrt{t'}}{1-t}X, 0, \frac{t'}{t'}\right) = \left(\frac{2}{\sqrt{1-t^2}}X, 0, 1\right)$$

and

$$\tilde{a}_{t'}(X,0,t) = C^{-1}a_{t'}C(X,0,t) = \frac{1}{4}\left(\frac{4}{\sqrt{1-t^2}}X, 0, 0\right) = \frac{1}{\sqrt{1-t^2}}X$$

as claimed. □

We now describe the geodesics of $B$. As one would expect, Theorem 4.9 below is consistent with Theorems 1.5 and 1.11.

**Theorem 4.9** *The geodesic through $p_1, p_2 \in \mathfrak{s}$, $p_1 \neq p_2$ is the circular arc through $p_1$ and $p_2$ contained in*

$$\Sigma = p_1 + \left(T^{(2)}_{p_2-p_1} \oplus \mathbf{R}(p_2 - p_1)\right)$$

*which is orthogonal to $\Sigma \cap S_{\mathfrak{s}}$.*

**Proof** This result has already been established in the case when $\mathfrak{v} = 0$, for then $\Sigma = \mathfrak{z} \oplus \mathfrak{a}$ and $\mathfrak{s}$ is isomorphic to $B^n$ with the Poincaré metric, where $n = \dim(\mathfrak{z} \oplus \mathfrak{a})$. We therefore assume that $\mathfrak{v} \neq 0$. There exists $k \in K$ such that $k(p_2 - p_1) = X \in \mathfrak{v}$. Let $\Pi = kp_1 + (\mathbf{R}X \oplus \mathfrak{j}(X))$. By linearity, if $w \in \Pi$ such that $w \perp (\mathbf{R}X \oplus \mathfrak{j}(X))$, then $\Pi = w + (\mathbf{R}X \oplus \mathfrak{j}(X))$. Let $k'$ be as in Lemma 4.7. By transitivity, there exists $m \in M$ such that $mk'w = k'(tH)$ for some $t \in \mathbf{R}$. The composition $(k')^{-1}mk'$ sends $\mathbf{R}X$ and $\mathfrak{j}(X)$ to themselves and $w$ to $tH$, hence $\Pi' = (k')^{-1}mk'(\Pi) = tH + (\mathbf{R}X + \mathfrak{j}(X))$.

Now let $t' = (1-t)/(1+t)$. By Lemma 4.8,

$$\tilde{a}_{t'}(tH + sX + J_{Z'}X) = \frac{1}{\sqrt{1-t^2}}(sX + J_{Z'}X)$$

for all $s \in \mathbf{R}$, $Z' \in \mathfrak{z}$. It follows that $k'\tilde{a}_{t'}\Pi' = \mathfrak{z} \oplus \mathfrak{a}$. We claim that $B_{\mathfrak{z}\oplus\mathfrak{a}}$ is totally geodesic in $B_{\mathfrak{s}}$. To see this, note that the isometry $(\Theta^{-1}C)$ maps $B_{\mathfrak{z}\oplus\mathfrak{a}}$ onto $\{0\} \times \mathfrak{z} \times \mathbf{R}^+ \subset S$; here $C$ is the Cayley transform and $\Theta : \mathfrak{s} \to \mathfrak{s}$ is the map

$$\Theta(X, Z, t) = (X, Z, t + \frac{1}{4}|X|^2)$$

for all $(X, Z, t) \in \mathfrak{v} + \mathfrak{z} + \mathfrak{a}$ as given in Section 3.1.2. By an argument similar to the proof of Proposition 2.1 of [CDKR2], we need only show that

$$\langle [T, U], U \rangle = 0$$

for all $U = (0, Z, t)$, $T = (X, 0, 0)$, with $Z \in \mathfrak{z}$, $X \in \mathfrak{v}$ and $t > 0$. This holds, for

$$\langle [T, U], U \rangle = \left\langle \frac{1}{2}tX, (0, Z, t) \right\rangle = 0.$$

The restriction of the metric on $B_{\mathfrak{s}}$ to $B_{\mathfrak{z}\oplus\mathfrak{a}}$ is the Poincaré metric on $B_{\mathfrak{z}\oplus\mathfrak{a}}$, for if $p \in \mathfrak{z} \oplus \mathfrak{a}$ then $T^{(2)}_p \oplus \mathbf{R}p = \mathfrak{z} \oplus \mathfrak{a}$. It follows that the geodesic through $k''p_1$ and $k''p_2$ is the circular arc through these points intersecting $S_{\mathfrak{z}\oplus\mathfrak{a}}$ orthogonally, where

$$k'' = k'\tilde{a}_{t'}(k')^{-1}mk'k.$$

Since $k'$, $k$ and $m$ are all orthogonal maps which satisfy

$$f\left(T_p^{(2)} + \mathbf{R}p\right) = T_{f(p)}^{(2)} + \mathbf{R}f(p)$$

for all $p \in B_{\mathfrak{s}}$, $f = k', k, m$, and also since $\tilde{a}_{t'} : tH + (X \oplus \mathfrak{j}(X)) \to X \oplus \mathfrak{j}(X)$ is effectively a scaling by $(1-t^2)^{-1/2}$, the result follows. □

We now establish the asymmetric distance formula.

**Theorem 4.10** *The distance function on $B_{\mathfrak{s}}$ is given by*

$$d(p_1, p_2) = 2\cosh^{-1}\left(\frac{\left| |p_2|\, P_{p_2} p_1 - \dfrac{p_2}{|p_2|}\right|}{\sqrt{1-|p_1|^2}\sqrt{1-|p_2|^2}}\right)$$

*for all $p_1, p_2 \in B_{\mathfrak{s}}$, $p_2 \neq 0$.*

**Proof** We first consider the case when $\mathfrak{v} = 0$. In this case, since $T_{p_2}^{(2)} \oplus \mathbf{R}p_2 = \mathfrak{z} \oplus \mathfrak{a}$, we have $P_{p_2} p_1 = p_1$. Now $\mathfrak{s}$ is isomorphic to $B^n$ with the Poincaré metric, where $n = \dim(\mathfrak{z} \oplus \mathfrak{a})$. Using the identity

$$\cosh^{-1}(1+2a) = 2\cosh^{-1}\sqrt{1+a}$$

which holds for all real $a$, we have

$$\begin{aligned}
d(p_1, p_2) &= \cosh^{-1}\left(1 + \frac{2\,|p_1 - p_2|^2}{(1-|p_1|^2)(1-|p_2|^2)}\right) \\
&= 2\cosh^{-1}\sqrt{1 + \frac{|p_1|^2 + |p_2|^2 - 2\langle p_1, p_2\rangle}{(1-|p_1|^2)(1-|p_2|^2)}} \\
&= 2\cosh^{-1}\sqrt{\frac{1 - 2\langle p_1, p_2\rangle + |p_1|^2\,|p_2|^2}{(1-|p_1|^2)(1-|p_2|^2)}} \\
&= 2\cosh^{-1}\frac{\left| |p_2|\, p_1 - \dfrac{p_2}{|p_2|}\right|}{\sqrt{1-|p_1|^2}\sqrt{1-|p_2|^2}}
\end{aligned}$$

as required.

Now assume $\mathfrak{v} \neq 0$. Let $w, X, t, k, k', m, t', k''$ be as in the proof of the previous theorem. We show that $d$ is preserved under $f = k, k', m, k''$. For such $f$,

$|f(p_1)| = |p_1|$, $|f(p_2)| = |p_2|$ and $\langle f(p_1), f(p_2)\rangle = \langle p_1, p_2\rangle$ by the orthogonality of $f$. Furthermore, since $P_{f(p_2)} f(p_1) = f(P_{p_2} p_1)$,

$$\left|P_{f(p_2)} f(p_1)\right| = |f(P_{p_2} p_1)| = |P_{p_2} p_1|$$

again by the orthogonality of $f$, therefore $d$ is indeed preserved under $f$. Now the expression for $d$ is correct when $p_1, p_2 \in \mathfrak{z} \oplus \mathfrak{a}$ as seen above, thus to complete the proof, we need only show that

$$d((X_1, 0, t), (X_2, 0, t)) = d\left(\left(1 - t^2\right)^{-1/2} X_1, \left(1 - t^2\right)^{-1/2} X_2\right),$$

that is, $d|_{tH+\mathfrak{v}}$ is preserved under $\tilde{a}_{t'}$. Since

$$\{(X_1, 0, t), (X_2, 0, t)\}_{X_3} = \{X_1, X_2\}_{X_3} = 0$$

for any $X_3 \in \mathfrak{v}\backslash\{0\}$, Theorem 4.6 implies that we need only show that

$$\frac{\left(1 - t^2 - \langle X_1, X_2\rangle\right)^2 + \|[X_1, X_2]\|^2}{\left(1 - t^2 - |X_1|^2\right)\left(1 - t^2 - |X_2|^2\right)} = \frac{\left(1 - \langle \lambda X_1, \lambda X_2\rangle\right)^2 + \|[\lambda X_1, \lambda X_2]\|^2}{\left(1 - |\lambda X_1|^2\right)\left(1 - |\lambda X_2|^2\right)},$$

where $\lambda = \left(1 - t^2\right)^{-1/2}$. This equation follows trivially by linearity. $\square$

Note that Theorem 4.6 may be used to give the symmetric form of this formula, namely

$$d(p_1, p_2) = 2\cosh^{-1}\left(\frac{|(1 - \langle p_1, p_2\rangle)H + P_{\mathfrak{z}}[p_1, p_2] + \{p_1, p_2\}_p|}{\sqrt{1 - |p_1|^2}\sqrt{1 - |p_2|^2}}\right)$$

where $p = p_1$ or $p_2$ if $X_1, X_2 \neq 0$, or $p \in \mathfrak{s}\backslash\{0\}$ is arbitrary if either $X_1, X_2 = 0$.

## 4.2 The Boundary

In this section we define a distance function on the boundary $S_{\mathfrak{s}}$ of $B$. The $J^2$ condition is crucial in showing that the function is indeed a distance function. In fact the function is not even symmetric if the $J^2$ condition does not hold. We also demonstrate that if the $J^2$ condition does hold then the Cayley transform $C : B \to D$, when extended to the boundary, is 1-quasiconformal with respect to a natural distance function on the boundary of $D$.

### 4.2.1 The Boundary of $B$

We begin with a technical inequality.

**Lemma 4.11** *If $p_1 \in \mathfrak{z} \oplus \mathfrak{a}$ and $p_2 \in \mathfrak{v}$, then*

$$|p_1 + p_2| \, |P_{p_1+p_2} p| \leq |p_1| \, |P_{p_1} p| + |p_2| \, |P_{p_2} p|$$

*for all $p \in \mathfrak{s}$.*

**Proof** If either of $p_1$ or $p_2$ is 0, the result clearly holds. Assume that $p_1, p_2 \neq 0$. Let $p = (X_1, Z_1, t_1)$ and $p_1 = (0, Z_2, t_2)$. By (4.2),

$$\begin{aligned}
|p_1 + p_2|^2 \, |P_{p_1+p_2} p|^2 &= \langle X_1, p_2 \rangle^2 + |[X_1, p_2]|^2 + 2 \left\langle (t_1 - J_{Z_1}) X_1, (t_2 - J_{Z_2}) p_2 \right\rangle \\
&\quad + \left( |Z_1|^2 + t^2 \right) \left( |Z_2|^2 + t_2^2 \right) \\
|p_1|^2 \, |P_{p_1} p|^2 &= \left( |Z_1|^2 + t_1^2 \right) \left( |Z_2|^2 + t_2^2 \right) \\
|p_2|^2 \, |P_{p_2} p|^2 &= \langle X_1, p_2 \rangle^2 + |[X_1, p_2]|^2 .
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left( |p_1| \, |P_{p_1} p| + |p_2| \, |P_{p_2} p| \right)^2 \\
&\quad = \langle X_1, p_2 \rangle^2 + |[X_1, p_2]|^2 + \left( |Z_1|^2 + t_1^2 \right) \left( |Z_2|^2 + t_2^2 \right) \\
&\qquad + 2 \sqrt{\left( \langle X_1, p_2 \rangle^2 + |[X_1, p_2]|^2 \right) \left( |Z_1|^2 + t_1^2 \right) \left( |Z_2|^2 + t_2^2 \right)} \\
&\quad = \langle X_1, p_2 \rangle^2 + |[X_1, p_2]|^2 + \left( |Z_1|^2 + t_1^2 \right) \left( |Z_2|^2 + t_2^2 \right) \\
&\qquad + 2 \sqrt{\langle (t_1 - J_{Z_1}) X_1, (t_2 - J_{Z_2}) p_2 \rangle^2 + |[(t_1 - J_{Z_1}) X_1, (t_2 - J_{Z_2}) p_2]|^2} \\
&\quad \geq \langle X_1, p_2 \rangle^2 + |[X_1, p_2]|^2 + \left( |Z_1|^2 + t_1^2 \right) \left( |Z_2|^2 + t_2^2 \right) \\
&\qquad + 2 \left\langle (t_1 - J_{Z_1}) X_1, (t_2 - J_{Z_2}) p_2 \right\rangle \\
&\quad = |p_1 + p_2|^2 \, |P_{p_1+p_2} p|^2
\end{aligned}$$

by Lemma 4.2. $\square$

**Theorem 4.12** *Define $d_b : S_{\mathfrak{s}} \times S_{\mathfrak{s}} \to \mathbf{R}$ by*

$$d_b(p_1, p_2) = 2\sqrt{2} \, |P_{p_2} p_1 - p_2|^{1/2}$$

*for all $p_1, p_2 \in S_{\mathfrak{s}}$. Then $d_b$ is a distance function on $S_{\mathfrak{s}}$.*

**Proof** If $p_1, p_2 \in S_{\mathfrak{s}}$, then

$$d_b(p_1, p_1) = 2\sqrt{2}\,|P_{p_1}p_1 - p_1|^{1/2} = 2\sqrt{2}\,|p_1 - p_1|^{1/2} = 0$$

as $p_1 \in T^{(2)}_{p_1} \oplus \mathbf{R}p_1$, whereas if $d_b(p_1, p_2) = 0$, then

$$P_{p_2}p_1 = p_2.$$

In this case, write $p_1 = p_1' \oplus p_1''$, where $p_1' \in T^{(2)}_{p_2} \oplus \mathbf{R}p_2$. It then follows that

$$P_{p_2}p_1 = p_1',$$

so $p_1' = p_2$. Then

$$1 = |p_1|^2 = |p_1'|^2 + |p_1''|^2 = |p_2|^2 + |p_1''|^2 = 1 + |p_1''|^2,$$

whence $p_1'' = 0$ and $p_1 = p_2$. Since $d_b$ is nonnegative, it is therefore positive definite. Also, the symmetry of $d_b$ follows from Theorem 4.6 and the anticommutativity of $[\cdot,\cdot]$ and $\{\cdot,\cdot\}_X$ (see Lemma 4.4). It remains to verify the triangle inequality.

Let $u, v, w \in S_{\mathfrak{s}}$. Using an isometry $k \in K$, which preserves $d_b$ by Theorem 4.10 and orthogonality, we may assume that $u = H$. Now writing $v = (v_2, Z_2, t_2)$ and $w = w_1 \oplus w_2 \oplus w_3$, with $w_1 \in \mathfrak{z} \oplus \mathfrak{a}$ and $w_2 \in \mathbf{R}v_2 \oplus \mathfrak{j}(v_2) = \mathbf{R}v_2 \oplus T^{(2)}_{v_2}$, we see that

$$\frac{1}{8}(d_b(u, v))^2 = |u - P_u v| = |u - v_1|$$

where $v_1 = (Z_2, t_2) \in \mathfrak{z} \oplus \mathfrak{a}$, and

$$\frac{1}{8}(d_b(u, w))^2 = |u - P_u w| = |u - w_1|.$$

Lemma 4.11, with $p_1 = v_1$, $p_2 = v_2$ and $p = u - w$ gives

$$\begin{aligned} |P_v(u - w)| &\le |v_1|\,|P_{v_1}(u - w)| + |v_2|\,|P_{v_2}(u - w)| \\ &= |v_1|\,|u - w_1| + |v_2|\,|w_2| \end{aligned}$$

since $P_{v_1} = P_{\mathfrak{z}\oplus\mathfrak{a}}$, $P_{v_2}u = 0$ and $P_{v_2}w = w_2$. Next, note that

$$1 - |v_1|^2 \le 2(1 - |v_1|) = 2(|u| - |v_1|) \le 2\,|u - v_1|$$

by the triangle inequality for the inner product space $\mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, and similarly

$$1 - |w_1|^2 \le 2\,|u - w_1|.$$

It follows that

$$\begin{aligned}
|v_2|^2\,|w_2|^2 &= \left(1-|v_1|^2\right)\left(1-|w_1|^2-|w_3|^2\right) \\
&\leq \left(1-|v_1|^2\right)\left(1-|w_1|^2\right) \\
&\leq 4\,|u-v_1|\,|u-w_1| \\
&= \frac{1}{16}(d_b(u,v))^2(d_b(u,w))^2.
\end{aligned}$$

By the triangle inequality for $\mathfrak{v}\oplus\mathfrak{z}\oplus\mathfrak{a}$ and the fact that $|v_1|\leq 1$, we have

$$\begin{aligned}
\frac{1}{8}(d_b(v,w))^2 &= |P_v w - v| \\
&\leq |v-P_v u| + |P_v u - P_v w| \\
&= |v-P_v u| + |P_v(u-w)| \\
&\leq \frac{1}{8}(d_b(u,v))^2 + |v_1|\,|u-w_1| + |v_2|\,|w_2| \\
&\leq \frac{1}{8}(d_b(u,v))^2 + \frac{1}{8}(d_b(u,w))^2 + \frac{1}{4}(d_b(u,v))(d_b(u,w)) \\
&= \frac{1}{8}(d_b(u,v)+d_b(u,w))^2,
\end{aligned}$$

establishing the triangle inequality for $d_b$, hence $d_b$ is indeed a distance function on $S_{\mathfrak{s}}$. □

Since the preceding proof requires the $J^2$ condition, it is natural to ask whether the function $d_b$ is a distance function in the case of $H$-type algebras not satisfying this condition. In fact $d_b$ is not even symmetric if the $J^2$ condition does not hold.

**Theorem 4.13** *Let $\mathfrak{n}=\mathfrak{v}\oplus\mathfrak{z}$ denote an $H$-type algebra and $\mathfrak{s}=\mathfrak{n}\oplus\mathfrak{a}$ where $\mathfrak{a}$ is one-dimensional. Assume that $\mathfrak{n}$ does not satisfy the $J^2$ condition. Then the map $d_b : S_{\mathfrak{s}}\times S_{\mathfrak{s}}\to\mathbf{R}$ defined in the statement of Theorem 4.12 is not symmetric.*

**Proof** Since

$$|P_{p_1}p_2 - p_1|^2 = |P_{p_1}p_2|^2 + |p_1|^2 - 2\langle P_{p_1}p_2, p_1\rangle = |P_{p_1}p_2|^2 + 1 - 2\langle p_2, p_1\rangle,$$

the function $d_b$ is symmetric if and only if

$$|P_{p_1}p_2|^2 = |P_{p_2}p_1|^2$$

for all $p_1, p_2 \in S_{\mathfrak{s}}$.

Since the $J^2$ condition does not hold, we can find orthonormal vectors $Z_1, Z_2 \in \mathfrak{z}$ and $X \in \mathfrak{v}$ such that $J_{Z_1}J_{Z_2}X \notin \mathbf{R}X \oplus \mathfrak{j}(X)$. As

$$\langle J_{Z_1}J_{Z_2}X, X\rangle = -\langle J_{Z_1}X, J_{Z_2}X\rangle = -\langle Z_1, Z_2\rangle \langle X, X\rangle = 0,$$

we may set

$$J_{Z_1}J_{Z_2}X = kJ_{Z_3}X + Y,$$

where $Z_3 \in \mathfrak{z}$ is a unit vector, $0 \leq k < 1$ and $Y \in (\mathbf{R}X \oplus \mathfrak{j}(X))^{\perp}$. Then $Z_3 \perp Z_1, Z_2$, since

$$\langle J_{Z_1}J_{Z_2}X, J_{Z_1}X\rangle = \langle J_{Z_2}X, X\rangle = 0 \text{ and } \langle J_{Z_1}J_{Z_2}X, J_{Z_2}X\rangle = 0.$$

Extend $Z_1, Z_2, Z_3$ to an orthonormal basis $\{Z_i\}_{i=1}^q$ for $\mathfrak{z}$. Set $p_1 = c(X + H)$ and $p_2 = c(J_{Z_2}X + Z_1)$, where $c = 2^{-1/2}$. Then

$$\mathbf{R}p_1 \oplus T_{p_1}^{(2)} = \{((u + J_W)X, W, u) : u \in \mathbf{R}, W \in \mathfrak{z}\}$$

for which an (ordered) orthonormal basis is given by

$$\{c(X, 0, 1), c(J_{Z_1}X, Z_1, 0), c(J_{Z_2}X, Z_2, 0), \dots, c(J_{Z_q}X, Z_q, 0)\}.$$

It is clear that $p_2$ is orthogonal to all but the second and third vectors of this basis, giving

$$P_{p_1}p_2 = \frac{1}{2\sqrt{2}}(J_{Z_1}X + J_{Z_2}X + Z_1 + Z_2).$$

In particular,

$$|P_{p_1}p_2|^2 = \frac{1}{2}.$$

On the other hand,

$$\mathbf{R}p_2 \oplus T_{p_2}^{(2)} = \{((u + J_W)(-J_{Z_1})J_{Z_2}X, W, u) : u \in \mathbf{R}, W \in \mathfrak{z}\}$$

for which an (ordered) orthonormal basis is given by

$$\{c(-J_{Z_1}J_{Z_2}X, 0, 1), c(J_{Z_2}X, Z_1, 0), c(-J_{Z_2}J_{Z_1}J_{Z_2}X, Z_2, 0), v_3, \dots, v_q\}$$

where $v_i = c(-J_{Z_i}J_{Z_1}J_{Z_2}X, Z_i, 0)$ for $i = 3, \dots, q$. We denote the first three vectors of this basis by $v_0$, $v_1$ and $v_2$ respectively. Since

$$\langle -J_{Z_1}J_{Z_2}X, X\rangle = \langle J_{Z_2}X, J_{Z_1}X\rangle = \langle Z_2, Z_1\rangle \langle X, X\rangle = 0,$$

$$\langle J_{Z_2}X, X\rangle = 0 \quad \text{and} \quad \langle -J_{Z_2}J_{Z_1}J_{Z_2}X, X\rangle = \langle J_{Z_1}J_{Z_2}X, J_{Z_2}X\rangle = 0,$$

we see that $p_1$ is orthogonal to $v_1$ and $v_2$ and that $\langle v_0, p_1\rangle = \frac{1}{2}$. Furthermore, for $i = 3, \dots, q$, we have

$$\begin{aligned}\langle v_i, p_1\rangle &= \frac{1}{2}\langle -J_{Z_i}J_{Z_1}J_{Z_2}X, X\rangle = -\frac{1}{2}\langle J_{Z_i}(kJ_{Z_3}X + Y), X\rangle \\ &= \frac{1}{2}\langle kJ_{Z_3}X + Y, J_{Z_i}X\rangle = \frac{k}{2}\delta_{i3}.\end{aligned}$$

It follows that

$$P_{p_2}p_1 = \frac{1}{2\sqrt{2}}(-J_{Z_1}J_{Z_2}X - kJ_{Z_3}J_{Z_1}J_{Z_2}X + kZ_3 + H),$$

so that

$$|P_{p_2}p_1|^2 = \frac{1}{8}(2 + 2k^2) < \frac{1}{2}.$$

In particular,

$$|P_{p_2}p_1| \neq |P_{p_1}p_2|,$$

so $d_b$ is not symmetric. □

### 4.2.2 The Boundary of $D$ and the Cayley Transform

In $\mathfrak{v} \times \mathfrak{z} \times \mathfrak{a}$, the set

$$\partial D_0 = \left\{(X, Z, t) : t = \frac{1}{4}|X|^4\right\}$$

is evidently the boundary of $D$. The Cayley transform $C$ defined in (3.14) extends to a bijection $\partial B \setminus \{H\} \to \partial D_0$ given by the same formula. In order to define $C$ at the point $H \in \partial B$, we denote the one point compactification of $\partial D_0$ by $\partial D = \partial D_0 \cup \{\infty\}$ and define $C(H) = \infty$. The map $\sigma$ defined in (3.16) extends to a map $\partial D \to \partial D$ given by the same expression and the conditions $\sigma(e) = \infty$ and $\sigma(\infty) = e$, where $e = (0,0,0)$. We now identify $\partial D_0$ with $N \cong \mathfrak{v} \times \mathfrak{z}$ by restricting the map $(X, Z, t) \mapsto (X, Z)$. The image of $e$, also denoted by $e$, is the identity element of $N$. We identify $\partial D$ with $N \cup \{\infty\}$ in the same way. The boundary extension of $\sigma$ is now regarded as a map on $N \cup \{\infty\}$ satisfying

$$\sigma(e) = \infty, \qquad \sigma(\infty) = e \qquad \text{and}$$

$$\sigma(X,Z) = \mathcal{B}^{-1}\left(\left(-\frac{|X|^2}{4} + J_Z\right)X, -Z\right) \tag{4.3}$$

for all $(X, Z) \neq (0, 0)$, where

$$\mathcal{B}(X,Z) = \frac{|X|^4}{16} + |Z|^2.$$

Similarly we regard the extended Cayley transform $C$ as a map from $\partial B$ to $N \cup \{\infty\}$ given by

$$C(0,0,1) = C(H) = \infty$$

and

$$C(X,Z,t) = \frac{1}{(1-t)^2 + |Z|^2}(2(1-t+J_Z)X, 2Z)$$

for all $(X, Z, t) \in \partial B \backslash \{H\}$.

The Campbell-Baker-Hausdorff formula shows that multiplication in $N$ is given by

$$(X,Z)(X',Z') = \left(X + X', Z + Z' + \frac{1}{2}[X, X']\right)$$

for all $(X, Z), (X', Z') \in \mathfrak{v} \times \mathfrak{z}$. Equip $N$ with the left-invariant distance $d_N$ defined by

$$d_N(e,(X,Z)) = \left(\frac{|X|^4}{16} + |Z|^2\right)^{1/4} = (\mathcal{B}(X,Z))^{1/4}.$$

We may extend this distance to $N \cup \{\infty\}$ by setting $d(x, \infty) = \infty$ for all $x \in N$. We shall show that the extended Cayley transform is a 1-quasiconformal map with respect to the distances $d_b$ and $d_N$ on $\partial B$ and $N \cup \{\infty\}$ respectively. The map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces $X$ and $Y$ is said to be $\lambda$-*quasiconformal* at the point $x \in X$, where $\lambda > 1$, if for all $\varepsilon > 0$ and all sufficiently small $r > 0$, there exists $R > 0$ such that

$$B(f(x), R) \subseteq f(B(x,r)) \subseteq B(f(x), (\lambda + \varepsilon)R).$$

That is, the image of a small ball centred at $x$ is contained within two balls centred at $f(x)$, the ratio of whose radii is nearly bounded by $\lambda$. The map $f$ is said to be $\lambda$-quasiconformal if it is $\lambda$-quasiconformal at all $x \in X$. We extend this definition in the case where $X$ or $Y$ is $N \cup \{\infty\}$ in the following way. The map $f$ is said to be $\lambda$-quasiconformal at $\infty$ if $f\sigma$ is $\lambda$-quasiconformal at $e$. We use the map $\sigma$ in a similar way to define $\lambda$-quasiconformality at a point $x$ such that $f(x) = \infty$.

In order to prove the 1-quasiconformality of $C$, we first demonstrate that the inversion $\sigma : N \cup \{\infty\} \to N \cup \{\infty\}$ is 1-quasiconformal with respect to $d_N$. In fact Theorem 5.1 of [CDKR] asserts (amongst other things) the equivalence of the $J^2$ condition holding and the $\lambda$-quasiconformality of $\sigma$, for some $\lambda \geq 1$. The following lemma shows that we may take $\lambda = 1$.

**Lemma 4.14** *The map $\sigma : (N \cup \{\infty\}, d_N) \to (N \cup \{\infty\}, d_N)$ defined in (4.3) above is 1-quasiconformal.*

**Proof** That $\sigma$ is 1-quasiconformal at the points $e$ and $\infty$ is obvious from the definition and the fact that $\sigma$ is an involution. The lemma now follows from formula (3.3) in [CDKR] which holds whenever the $J^2$ condition holds:

$$\mathcal{B}\left(\sigma(n)^{-1}\sigma(n')\right) = \mathcal{B}(n)^{-1}\mathcal{B}(n')^{-1}\mathcal{B}\left(n^{-1}n'\right) \tag{4.4}$$

for all $n, n' \in N^* = N\backslash\{e\}$. In terms of the distance $d_N$, taking fourth roots of this equation gives

$$d_N(\sigma(n), \sigma(n')) = \frac{d_N(n, n')}{d_N(n, e)d_N(n', e)}$$

for all $n, n' \in N^*$. If we fix $n \in N^*$ with $d_N(n, e) = K$, and allow $n'$ to vary subject to $d_N(n, n') = \varepsilon$ where $\varepsilon < K$, then by the triangle inequality we have

$$\frac{\varepsilon}{K(K+\varepsilon)} \leq d_N(\sigma(n), \sigma(n')) \leq \frac{\varepsilon}{K(K-\varepsilon)}$$

which clearly establishes the lemma.

In order to prove (4.4) we follow the proof of Theorem 4.2 of [CDKR]. In particular, let

$$\mathcal{A}(X, Z) = \frac{|X|^2}{4} + J_Z \qquad \text{and} \qquad \overline{\mathcal{A}}(X, Z) = \frac{|X|^2}{4} - J_Z.$$

Then

$$\mathcal{A}(X, Z)\,\overline{\mathcal{A}}(X, Z) = \overline{\mathcal{A}}(X, Z)\,\mathcal{A}(X, Z) = \mathcal{B}(X, Z)$$

and

$$\sigma(X, Z) = (\mathcal{B}(X, Z))^{-1}(-\overline{\mathcal{A}}X, -Z)$$

for all $(X, Z) \in N^*$. Now if $n = (X, Z)$, $n' = (X', Z')$ and $\mathcal{A}$, $\mathcal{A}'$, $\overline{\mathcal{A}}$, $\overline{\mathcal{A}}'$, $\mathcal{B}$, $\mathcal{B}'$ are abbreviations for $\mathcal{A}(X, Z)$, $\mathcal{A}(X', Z')$, $\overline{\mathcal{A}}(X, Z)$, $\overline{\mathcal{A}}(X', Z')$, $\mathcal{B}(X, Z)$, $\mathcal{B}(X', Z')$ respectively, a simple calculation (see [CDKR] for details) gives

$$\begin{aligned}\mathcal{B}(n^{-1}n') = \mathcal{B}' + \mathcal{B} &+ \frac{1}{4}(\langle X, X'\rangle^2 + |[X, X']|^2 \\ &+ 2\left(\frac{|X|^2\,|X'|^2}{16} - \langle Z, Z'\rangle\right) - \left\langle \overline{\mathcal{A}}'X', X\right\rangle - \langle \overline{\mathcal{A}}X, X'\rangle .\end{aligned}$$

(Note that the conjugation bars over $\mathcal{A}$ and $\mathcal{A}'$ were incorrectly omitted in formula (4.5) of [CDKR].) We now apply this formula to $\sigma(n)$ and $\sigma(n')$ in place of $n$ and $n'$ respectively, and use the formulae $\mathcal{B}(\sigma(n)) = \mathcal{B}(n)^{-1}$, $\left|\mathcal{B}^{-1}\overline{\mathcal{A}}X\right|^2 = \mathcal{B}^{-1}\,|X|^2$ and

$$\overline{\mathcal{A}}(\sigma(X, Z))\left(-\mathcal{B}^{-1}\overline{\mathcal{A}}X\right) = -\mathcal{B}^{-1}X,$$

which are easily established. We obtain

$$\begin{aligned}&\mathcal{B}(\sigma(n)^{-1}\sigma(n')) \\ &\quad = \mathcal{B}^{-1}(\mathcal{B}')^{-1}\Bigg(\mathcal{B}' + \mathcal{B} + \frac{\mathcal{B}^{-1}(\mathcal{B}')^{-1}}{4}\left(\left\langle \overline{\mathcal{A}}X, \overline{\mathcal{A}}'X'\right\rangle^2 + \left|[\overline{\mathcal{A}}X, \overline{\mathcal{A}}'X']\right|^2\right) \\ &\qquad +2\left(\frac{|X|^2\,|X'|^2}{16} - \langle Z, Z'\rangle\right) - \left\langle \overline{\mathcal{A}}'X', X\right\rangle - \langle \overline{\mathcal{A}}X, X'\rangle\Bigg).\end{aligned}$$

Formula (4.4) now follows from Lemma 4.2. □

Recall that $G$ denotes the full group of isometries of $D$.

**Lemma 4.15** *Let $G_0$ denote the connected component of the identity of $G$. Let $G_1$ denote the subgroup of $G$ generated by $N$ and $\sigma$. Then $G_0 \subseteq G_1$.*

**Proof** It suffices to show that there is no proper Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ containing $\mathfrak{n}$ and $\theta\mathfrak{n}$, where $\theta : \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity of $G$ of the involution $g \mapsto \sigma g \sigma$. Proposition 4.7(ii) of [CDKR2] asserts that $[X, \theta X] = -|X|^2 H$ for all $X \in \mathfrak{v}$, whence $\mathfrak{g}'$ contains $\mathfrak{a}$. By Lemma 4.1(iii) of [CDKR2], $\mathfrak{p} = \mathfrak{a} \oplus (I - \theta)\mathfrak{n}$ where $\mathfrak{p}$ is the $-1$ eigenspace of $\theta$. This implies that $\mathfrak{p} \subset \mathfrak{g}'$. In the proof of Theorem 4.6 of [CDKR2] it is established that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of $K$, the stabiliser subgroup of $(0, 0, 1) \in D$. (In fact, $\mathfrak{k}$ is the $+1$ eigenspace of $\theta$.) This implies that $\mathfrak{k} \subset \mathfrak{g}'$. Finally, since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we conclude that $\mathfrak{g}' = \mathfrak{g}$ as claimed. □

**Corollary** *The group $G_0$ acts 1-quasiconformally on $(N \cup \{\infty\}, d_N)$.*

**Proof** By Lemma 4.14, $\sigma$ is 1-quasiconformal. Furthermore $N$ acts isometrically on itself with respect to $d_N$. Since isometries are 1-quasiconformal and any composition of 1-quasiconformal maps is itself 1-quasiconformal, the corollary follows. $\square$

Recall that we use the Cayley transform to transfer isometries of $D$ to isometries of $B$, that is,

$$\tilde{g} = C^{-1} g C$$

for any isometry $g : D \to D$.

**Lemma 4.16** *Let $G_0$ denote the connected component of the identity of $G$. Then the group $K \cap G_0$ acts transitively on the boundary $N \cup \{\infty\}$.*

**Proof** By the remarks preceding the statement of this lemma, we may instead establish that $K_0 = (K \cap G_0)^{\tilde{}}$ acts transitively on $\partial B$. Fix a point $p \in \partial B$. Since $K$ is compact and $G_0$ is closed, $K_0$ is compact, therefore the orbit of $p$ is closed in $\partial B$. Since $\tilde{K}$ is a finite cover of $K_0$ and the orbit of $p$ under $\tilde{K}$ is the whole of $\partial B$, we conclude that the orbit of $p$ is open in $\partial B$, hence is the whole of $\partial B$. $\square$

**Theorem 4.17** *The extended Cayley transform $C : (\partial B, d_b) \to (N \cup \{\infty\}, d_N)$ is 1-quasiconformal.*

**Proof** It suffices to demonstrate the 1-quasiconformality of $C$ at the point $-H$ of $\partial B$; for suppose that $C$ is indeed 1-quasiconformal there. Let $p \in \partial B$ be arbitrary. By Lemma 4.16, there exists $k \in K \cap G_0$ such that $\tilde{k}(-H) = p$. Furthermore $\tilde{k}$, considered as a map from $\partial B$ to itself, is an isometry with respect to $d_b$. (This follows from the orthogonality of $\tilde{k}$ and the relationship between the interior distance $d$ (on $B$) and $d_b$.) Since $C = kC\tilde{k}^{-1}$, $\tilde{k}^{-1}$ is 1-quasiconformal at $p$, $C$ is 1-quasiconformal at $-H$ and $k$ is 1-quasiconformal at $e \in N \cong \partial B$ (by the corollary to Lemma 4.15), we conclude that $C$ is 1-quasiconformal at $p$.

We therefore need only check the 1-quasiconformality of $C$ at $q = -H \in \partial B$. Fix $\varepsilon > 0$ and take $p = (X, Z, t) \in \partial B$ such that $d_b(p, q) = \varepsilon$. We have

$$\begin{aligned}
\varepsilon = d_b(p,q) &= 2\sqrt{2}\,|P_q p - q|^{1/2} \\
&= 2\sqrt{2}\,|(0, Z, t) - (0, 0, -1)|^{1/2} \\
&= 2\sqrt{2}\big(|Z|^2 + (t+1)^2\big)^{1/4},
\end{aligned}$$

or

$$|Z|^2 = \frac{\varepsilon^4}{64} - (t+1)^2 = 4\delta^4 - (t+1)^2,$$

where we have made the substitution $\delta = \varepsilon/4$. Let

$$(X_1, Z_1) = C(p) = \frac{2}{(1-t)^2 + |Z|^2}((1-t+J_Z)X, Z).$$

We have

$$d = d_N(C(p), C(q)) = \left(\frac{1}{16}|X_1|^4 + |Z|^2\right)^{1/4}$$

since $C(q) = (0,0)$. Since $|X|^2 + |Z|^2 + t^2 = 1$,

$$\begin{aligned}
|X_1|^4 &= \frac{16\left((1-t)^2 + |Z|^2\right)^2 |X|^4}{\left((1-t)^2 + |Z|^2\right)^4} \\
&= \frac{16\left(1-t^2-|Z|^2\right)^2}{\left((1-t)^2 + |Z|^2\right)^2} \\
&= \frac{16(1-t^2-4\delta^4+(1+t)^2)^2}{((1-t)^2+4\delta^4-(1+t)^2)^2} \\
&= \frac{16(2(1+t)-4\delta^4)^2}{(4\delta^4-4t)^2}
\end{aligned}$$

while

$$\begin{aligned}
|Z_1|^2 &= \frac{4|Z|^2}{\left((1-t)^2+|Z|^2\right)^2} \\
&= \frac{4(4\delta^4-(t+1)^2)}{(4\delta^4-4t)^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
d^4 = \frac{1}{16}|X_1|^4 + |Z_1|^2 &= \frac{(2(1+t)-4\delta^4)^2 + 4(4\delta^4-(t+1)^2)}{(4\delta^4-4t)^2} \\
&= \frac{\delta^4}{\delta^4 - t}.
\end{aligned}$$

For $\varepsilon$ (hence $\delta$) sufficiently small, $t < 0$. From

$$|Z|^2 + t^2 \leq 1 \qquad \text{and} \qquad |Z|^2 + (t+1)^2 = 4\delta^4$$

we obtain

$$t^2 - (t+1)^2 \leq 1 - 4\delta^4,$$

or

$$-t \leq 1 - 2\delta^4.$$

Also

$$(t+1)^2 \leq 4\delta^4$$

gives

$$-t \geq 1 - 2\delta^2$$

since $t + 1 \geq 0$. Using these estimates, we see that

$$d^4 \leq \frac{\delta^4}{\delta^4 + (1 - 2\delta^2)} = \frac{\delta^4}{(1-\delta^2)^2}$$

and

$$d^4 \geq \frac{\delta^4}{\delta^4 + (1 - 2\delta^4)} = \frac{\delta^4}{1-\delta^4}.$$

Since the ratio of these two quantities tends to 1 as $\delta \to 0$, the map $C$ is indeed 1-quasiconformal at $-H$. $\square$

# Appendix A

# Triality Calculations

In Section 2.4.4 we linked the triality automorphism of $Spin(8)$ with the Lie algebraic properties of $\mathfrak{so}(8)$, in particular the roots thereof. In this appendix we present some of the calculations involved.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(8)$ of $G = SO(8)$ consists of all real skew-symmetric matrices. A basis is given by

$$\{X_{jk} : 0 \leq j < k \leq 7\}, \qquad X_{jk} = E_{jk} - E_{kj}.$$

A maximal torus $\mathfrak{t}$ is spanned by $\{Y_j\}_{j=0}^{3}$ where

$$Y_j = X_{2j,2j+1}$$

for $j = 0, 1, 2, 3$. If $H = \sum_{j=0}^{3} a_j Y_j \in \mathfrak{t}$ for reals $a_1, a_2, a_3, a_4$, then

$$\begin{aligned}
[H, X_{2j,2k}] &= -a_j X_{2j+1,2k} - a_k X_{2j,2k+1} \\
[H, X_{2j,2k+1}] &= -a_j X_{2j+1,2k+1} + a_k X_{2j,2k} \\
[H, X_{2j+1,2k}] &= a_j X_{2j,2k} - a_k X_{2j+1,2k+1} \\
[H, X_{2j+1,2k+1}] &= a_j X_{2j,2k+1} + a_k X_{2j+1,2k}
\end{aligned}$$

for all $0 \leq j < k \leq 3$, whereas $[H, Y_j] = 0$ for all $j = 0, 1, 2, 3$. This implies that the roots are given by

$$\{i(\pm Y_j^* \pm Y_k^*)\}_{0 \leq j < k \leq 3},$$

with root spaces spanned by

$$(X_{2j,2k} - X_{2j+1,2k+1}) \pm i(X_{2j,2k+1} + X_{2j+1,2k})$$

corresponding to the roots $\pm i(Y_j^* + Y_k^*)$ and

$$(X_{2j,2k} + X_{2j+1,2k+1}) \pm i(-X_{2j,2k+1} + X_{2j+1,2k})$$

corresponding to the roots $\pm i(Y_j^* - Y_k^*)$, for $0 \le j < k \le 3$.

Let the positive roots be given by

$$i(Y_j^* \pm Y_k^*), \qquad 1 \le j < k \le 4,$$

where $Y_4 = Y_0$ for convenience. (The subscripts in $X_{j,k}$ should be taken modulo 8 in order to comply with this renumbering.) An ordered basis is then given by

$$\{i(Y_2^* - Y_3^*), i(Y_3^* + Y_4^*), i(Y_3^* - Y_4^*), i(Y_1^* - Y_2^*)\},$$

denoted $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ respectively. The Dynkin diagram is depicted in Figure 2.1. Let $\Theta$ be the rotation of $i\mathfrak{t}^*$ defined by

$$\Theta(\alpha_0) = \alpha_0, \quad \Theta(\alpha_1) = \alpha_2, \quad \Theta(\alpha_2) = \alpha_3, \quad \Theta(\alpha_3) = \alpha_1$$

and extended by linearity. Clearly $\Theta$ is of order 3 and is given by

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$

with respect to the basis $\{iY_1^*, iY_2^*, iY_3^*, iY_4^*\}$. By a slight abuse of notation, we may regard $\Theta$ as a map on $\mathfrak{t}$ by defining it to be induced by the same matrix with respect to the basis $\{Y_1, Y_2, Y_3, Y_4\}$ of $\mathfrak{t}$. We wish to extend $\Theta$ to a Lie algebra automorphism of all of $\mathfrak{g}$. If $Z_\alpha$ is a vector in the root space $R_\alpha$ for the root $\alpha$ in the complexification of $\mathfrak{g}$, then $[H, Z_\alpha] = \alpha(H)Z_\alpha$ for all $H \in \mathfrak{t}$. As $\Theta$ is to be a Lie algebra automorphism,

$$[\Theta H, \Theta Z_\alpha] = \alpha(H)\Theta Z_\alpha = \left((\Theta^{-1})^* \alpha\right)(\Theta H)\Theta Z_\alpha = (\Theta\alpha)(\Theta H)\Theta Z_\alpha$$

since $\Theta$ is orthogonal. This implies that

$$\Theta(R_\alpha) = R_{\Theta\alpha}$$

for all roots $\alpha$. Clearly we need only ensure that this holds for all positive roots. For the positive roots $\alpha = i(Y_j^* + Y_k^*)$, $1 \le j < k \le 4$, define

$$X_\alpha = X_{2j,2k} - X_{2j+1,2k+1}, \qquad Y_\alpha = X_{2j,2k+1} + X_{2j+1,2k}$$

and for the positive roots $i(Y_j^* - Y_k^*)$, $1 \le j < k \le 4$, define

$$X_\alpha = s(X_{2j,2k} + X_{2j+1,2k+1}), \qquad Y_\alpha = t(-X_{2j,2k+1} + X_{2j+1,2k})$$

where $s = 1$ except when ($j = 1$ and $k = 3$) or ($k = 4$) in which case $s = -1$; and $t = 1$ except when $j = 1, k = 2$ in which case $t = -1$. It is easy to verify that in all cases,

$$R_\alpha = \mathrm{span}_{\mathbf{C}}\{X_\alpha + iY_\alpha\}.$$

Extend $\Theta$ to all of $\mathfrak{g}$ by requiring that

$$\Theta(X_\alpha) = X_{\Theta\alpha} \qquad \text{and} \qquad \Theta(Y_\alpha) = Y_{\Theta\alpha}$$

for all positive roots $\alpha$ and extending by linearity. It is easy to verify that $\Theta$ is a Lie algebra automorphism of order 3. In fact a calculation shows that

$$\Theta(X_{j,k}) = \frac{1}{2}\nu(e_k)\nu(e_j)$$

for all $0 \le j < k \le 7$, where $\nu$ is given in Section 2.4.1 and $\{e_0, \ldots, e_7\}$ is the standard basis of $\mathbf{R}^8$.

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