The strength and stiffness of rectangular reinforced concrete beams in combined bending and torsion

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THE STRENGTH AND STIFFNESS OF RECTANGULAR REINFORCED CONCRETE BEAMS IN COMBINED BENDING AND TORSION

by

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A thesis submitted to the Professorial Board of the University of New South Wales as partial fulfilment of the requirements for the Degree of Doctor of Philosophy.

THIS IS TO CERTIFY THAT THE CONTENTS OF THIS THESIS HAVE NOT BEEN SUBMITTED TO ANY OTHER UNIVERSITY OR INSTITUTION FOR A HIGHER DEGREE.

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ABSTRACT

This thesis reports the results of an investigation into, the ultimate strength of beams loaded in combined torsion, bending and shear and the deformations of members subjected to combined torsion and flexure.

Chapter 2 presents the results of a critical review of the published investigations into the behaviour of reinforced concrete members subjected to torsion with and without other actions. In addition the various strength theories which have been proposed are discussed and a comparison made between the published experimental results and the predictions of these theories.

The results of the experimental phase of this study are reported in Chapter 3. Tests were conducted on plain concrete specimens, beams reinforced in the longitudinal direction only and beams containing both longitudinal and transverse steel. The loads were applied to produce either pure torsion or torsion combined with flexure.

An empirical treatment of the problem of the ultimate strength of beams without web reinforcement is given in Chapter 4. A simplified design procedure is developed.

Chapter 5 presents an analysis method for the ultimate capacity of a web reinforced beam subjected to the combined loading. Equations are derived for the interaction behaviour in bending and torsion. In Chapter 6 the results of the tests of both this investigation and published works are used to verify the analysis methods for web reinforced beams.

The work on the ultimate strength of beams is completed by the design procedure set out in Chapter 7.

Finally a theory for the deflections and rotations of web reinforced beams at service loads is offered in Chapter 8.

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NOTATION

In addition to special symbols which are defined where they appear, the following symbols are used in this thesis;

A_{L1} = The cross-sectional area of longitudinal steel near the tension face of the beam subjected to flexure.

A_{L2} = The cross-sectional area of longitudinal steel near the side face of the beam.

A_{L3} = The cross-sectional area of longitudinal steel near the compression face of the beam subjected to flexure.

A_w = The cross-sectional area of one leg of hoop reinforcement.

a_1, a_2 = The cover on longitudinal steel.

a_3, a_4 = The cover to the hoops from the sides and bottom of the beam.

b = The width of the beam (minimum dimension).

b' = The width of the hoop.

d' = The height of the hoop.

d = The distance from the extreme compression fibre (for flexure) to the centroid of A_{L1}.

e_w = The strain in the web steel.

e_s = The strain in the longitudinal steel.

E_c = The modulus of elasticity for concrete.

f'_t = The tensile strength of concrete.

f'_c = The compressive strength of the concrete.

$f_{L1}, f_{L2}, f_{L3}, f_w$ = The yield strength of steels A_{L1}, A_{L2}, A_{L3} and A_W.

G = The shear modulus of rigidity.

h = The depth of the beam.

I_{cr} = The cracked transformed section moment of inertia.

I_g = The gross section moment of inertia.

I_{gt} = The gross transformed section moment of inertia.

jd = The lever arm of the longitudinal steel as computed from the modular ratio theory.

K_1, K_2 = Constants defined by the elastic theory for torsion.

M = The bending moment applied to the section.

M_u = The computed ultimate flexural capacity of the section in simple flexure.

M_{cr} = The cracking moment.

p' = $\dfrac{A_{L3}}{bd}$

p = $\dfrac{A_{L1}}{bd}$

R = The ratio of A_{L3} f_{L3} to A_{L1} f_{L1}.

r = A parameter relating transverse to longitudinal steel.

r_o = The design value of r.

s = The spacing of hoops.

T_1, T_2, T_3, T_{eff} = The predicted torsional strength in mode 1, mode 2, mode 3 and the effective shear mode respectively.

T_c = The torque resisted by the concrete.

T_s = The torque resisted by the steel.

T_{cr} = The cracking torque.

T_o = The pure torsional capacity of a beam.

T = The twisting moment applied to the section.

T' = $T \sqrt{1 + 2\alpha}$

V = The shear applied to the section.

V_o = The shear capacity of the section in simple flexure.

V_{eff} = An effective shear force ($V + 1.6\ T/b$).

x_1, x_2, x_3 = The depth of compression in mode 1, mode 2 and mode 3.

α = The ratio of the height to the width of the section.

β = The ratio of the effective height to the effective width of the section.

δ = $\frac{Vb}{2T}$

$\theta_1, \theta_2, \theta_3$ = The inclination of the compression hinge to the cross-section of the beam in mode 1, mode 2 and mode 3.

θ = Therotation per unit length of the beam.

θ_s = The rotation per unit length of the beam induced by steel strain.

θ_c = The rotation per unit length of the beam induced by concrete strains.

ψ = The ratio of torque to moment (T/M).

CHAPTER 1

INTRODUCTION

1.1 GENERAL

The structural requirements of a reinforced concrete member can be stated as, (i) an adequate reserve of strength against collapse and (ii) satisfactory performance at service loads. The logical method of meeting these requirements is the use of ultimate strength design procedures and when necessary calculations of deformations at working loads. It was the object of this thesis to develop suitable methods so that this approach could be extended to the case of combined torsion, bending and shear.

In particular a study was made into the failure behaviour and ultimate strength of rectangular reinforced concrete beams subjected to combined torsion, bending and shear. The deflections and rotations of web reinforced beams were also considered. A solution to these problems was sought that satisfied the dual criteria of accuracy and simplicity.

An extensive experimental program was undertaken to provide,
(i) the understanding of the failure behaviour essential to the development of the analysis equations,
(ii) ultimate strength test data for a wide range of parameters required for the confirmation of the proposed theories.

It should be noted that the sections of this thesis which deal with the ultimate strength of beams were carried out as a joint project with Collins (ref 1.1). The work on deflections and rotations was, however, an individual project.

1.2 LAYOUT AND SCOPE

In general, research is a process where new knowledge is built up step by step from existing work. Considerable attention was, therefore, devoted to a study of the literature on torsion in concrete. A critical discussion of previous investigations is given in Chapter 2. From this work it was found that although a surprising number of investigations had been conducted in this field very little of this work was sufficiently comprehensive for design use. Cowan and Armstrong provided the first satisfactory working stress design procedure for bending and torsion, but their work only partially answered the problems of bending torsion interaction and was, of course, restricted to working loads. Furthermore shear and torsion was not considered. The most important work on ultimate strength was carried out by Lessig, Chinenkov and Lyalin. Their method was complex and very restricted in its application. The ultimate equilibrium failure mechanisms were however of great importance and have affected all subsequent ultimate strength theories. Virtually no work has been done in the field of deflections and rotations in combined bending and torsion. The literature survey provided ideas that were developed and modified, topics for detailed study, and experimental data.

Even if only one beam section was chosen for study, a large number of tests would still be required to investigate the effect of the ratio of torsion to moment and the ratio of torsion to shear. Added to this is the need to consider plain concrete beams, beams reinforced in the longitudinal direction only and web reinforced beams. Further tests are then necessary to take into account factors such as the proportion of longitudinal to transverse steel, overall amount of reinforcement and span to depth ratio. In an attempt to encompass as many of these parameters as possible a large experimental program was undertaken.

The discussion of the experimental work is given in Chapter 3. To accomplish the desired test program instrumentation to the early test series was restricted. Strain gauge, rotation and deflection results are however available for beams of the deformation series.

The ultimate strength of beams containing only longitudinal reinforcement loaded in combined torsion, bending and shear is treated in Chapter 4. For this case an empirical approach was adopted. A more accurate solution to this problem does not appear justified until better methods are available for the related problems of shear strength of beams and tensile strength of concrete. The design procedure recommended, although simple, offers considerable improvements in accuracy and efficiency over current code rules.

Chapter 5 presents an analysis method for the ultimate strength of web reinforced beams loaded in torsion combined with bending and shear. The theory is based on the equilibrium of failure models. Particular emphasis is placed on the interaction of bending and torsion on the capacity of a member. In this thesis only an empirical equation is given for the loading combination of shear and secondary torsion. A more detailed treatment of this loading case is given by Collins.

The analysis method is extensively verified by comparison with experimental results in Chapter 6.

The analysis methods in Chapters 4, 5 and 6 are brought to their logical conclusion in the design procedures presented in Chapter 7.

Finally, a method which enables the deflections and rotations of web reinforced beams, at service loads, to be computed is developed in Chapter 8.

CHAPTER 2

HISTORICAL SURVEY

The object of this chapter is to present a critical review of the investigations carried out in torsion. Where possible, comparison with experimental results is used to evaluate the reported theories.

2.1 BEAMS WITHOUT WEB REINFORCEMENT

2.1 (a) Pure Torsion

As early as 1911 it was recognised that beams reinforced with longitudinal steel only and loaded in pure torsion fail immediately after the formation of the first diagonal tension crack. The twisting moment to cause such cracking is comparable with the maximum twisting moment which can be resisted by a plain concrete section similar to the reinforced member in all respects except for reinforcement. The ultimate torsional capacity has been reported as up to 15% greater than that for comparable plain concrete sections, although some tests (Young, Sagar and Hughes) (2.1) have shown lower strengths for the reinforced sections. This decrease is probably due to restrained shrinkage.

Longitudinal steel only is not effective in preventing the opening of diagonal tension cracks resulting from twisting moment, and it is inadvisable to consider such reinforcement as increasing the torsional capacity of the member. In other words, the torsional strength of a beam with longitudinal steel only should be assumed to be the same as a beam without reinforcement.

It is therefore of considerable interest to study the case of plain concrete beams. The strength of plain beams depends upon the stress distribution across the section and the failure criterion in concrete.

In the early investigations the classical elastic stress distribution was adopted. Some later reports recommend the use of the fully plastic stress distribution. In the plastic theory, concrete is assumed to behave as a rigid-plastic material. This results in a constant shear stress across the section. Very little proof of the truth of either of these theories is available. Marshall (2.2) chose the plastic theory on the basis of its consistency for groups of test results in which the same concrete was used. As the torsional shear stress-strain relationship exhibits a small amount of plasticity prior to concrete failure Kemp (2.3) recommended the plastic approach. In support of this conclusion he quotes theoretical work by Armstrong (2.4) that demonstrates that small plastic strains produce a relationship between maximum stress and torque that closely approximates the plastic theory. In both cases a formula for torque in terms of size and maximum tensile stress can be obtained.

1. Elastic Theory

$$T = K\, b^2\, h\, f_t' \qquad \ldots \qquad (2.1)$$

Where

$$K = \frac{1}{3 + \frac{2.6}{0.45} + h/b} \qquad \text{Approximately}$$

2. Plastic Theory

$$T = \frac{1}{2}\, b^2 \left(h - \frac{b}{3}\right) f_t' \qquad \ldots \qquad (2.2)$$

A far more significant problem is the failure criterion of concrete. As the failure of a plain specimen occurs with cleavage along a 45^{o} spiral it has reasonably been concluded that a maximum principal tensile stress criterion would be appropriate. The actual value of the limiting stress to be used with the above formula has been largely ignored. Results of tensile tests on the concrete are of limited use as a variety of tests have been employed and generally the strength obtained depends upon an elastic analysis of the test specimen, e.g. Brazilian test or Modulus of Rupture. Kemp suggests that a suitable value for the limiting stress to be used with the plastic theory is,

$$f'_t = 4\sqrt{f'_c} \qquad \ldots \qquad (2.3)$$

This approach is quite useful, as in general, concrete is specified by its compressive strength. However, some results indicate strengths rather below that given by the above formula.

2.1 (b) Beams Without Web Reinforcement Subjected To Combined Bending and Torsion

Except in very exceptional circumstances torsion will be accompanied by bending in reinforced concrete construction. Very frequently, when torsional moments are low the designer may elect to use a beam with longitudinal steel only. It is therefore important that information is available regarding the possible effects of bending on torsion and vice versa. Only three investigations appear to have been made; one by Nylander (2.5) in 1945, one by Gesund and Boston (2.6) in 1964 and one by Ramakrishnan and Vijarangan (2.7) in 1963. The ultimate strength of a beam of this type depends upon; the strength of the concrete in compression and tension combined with other stresses, crack propagation, dowel forces as a function of concrete or steel shear stresses and perhaps aggregate interlock. The problem is somewhat similar to combined shear and flexure in its difficulties.

Nylander and Gesund and Boston have proposed methods of treating two of the possible mechanisms of failure. It should be recognised that neither of these methods accounts for all possible modes of failure.

Nylander proposed a design method for normally reinforced beams subjected to bending and torsion. He analysed a cracked section resisting the combined forces and considered that torsion was resisted in two ways, partly by the uncracked concrete zone and partly by shear forces in the steel acting about the centre of the concrete compression zone. The torsion resisted wholly by the concrete he expressed in the form βT_o where T_o is the pure torsional capacity of the section as calculated from the plastic theory and β is a constant depending upon the shape of the section and the amount of longitudinal reinforcement. Nylander gives a table of the values of β.

e.g. $\beta = 0.55$ when $\frac{d}{b} = 1.5$ and $p = 0.005$

He thus computed the shearing stresses set up in the steel by the remaining torque. Additionally he calculated the direct stresses in the steel by the normal bending formula. Then, using the Huber-Beltrami failure criterion for combined shear and tension on the steel, he obtained the following formula for the amount of longitudinal steel required;

$$A_{L1} = \frac{1}{f_{L1}} \sqrt{\left(\frac{M}{jd}\right)^2 + 3\left(\frac{T - \beta\, T_o}{\frac{1}{2}\, d}\right)^2} \quad \ldots \quad (2.4)$$

This theory is based on two main assumptions, (i) that there exists an uncracked concrete zone and (ii) that the dowel forces are governed by yield of the steel. For beams tested with high ratios of torsion to bending a tension crack can cross the top surface before failure leaving no uncracked zone. Furthermore, the dowel forces are usually limited by the capacity of the beam to resist spalling of the concrete. However the mechanism proposed by Nylander does give a reasonable representation of the failure behaviour of a beam loaded with predominantly bending forces.

Gesund and Boston considered that the torsional component of the loading was resisted wholly by dowel forces exerted by the longitudinal bars. Moreover the dowel forces were limited by spalling of the concrete. The torsional capacity is only influenced by bending in as much as the magnitude of the dowel forces depends on the spacing of flexural cracks. By assuming that the dowel force on any bar, other than the bar causing spalling, is proportional to its distance from the longitudinal axis about which the beam rotates at failure, they obtained the following formula for torsional capacity:

$$T = F_c \left(r_c + \frac{1}{r_c}\, \Sigma r_i^{\,2}\right) \quad \ldots \quad (2.5)$$

where F_c is the dowel force on the critical bar, that is, the bar at which spalling occurs,

r_c is the distance of this bar from the failure hinge, and

r_i is the distance of the i^{th} bar from the failure hinge.

To find the value of F_c, it is necessary to calculate the force required to spall off a block of concrete. The method proposed by Gesund and Boston is a trial and error process and involves making assumptions, not easily justified, regarding the shape of the concrete spall, the spacing of the flexural cracks, the bond strength and the bearing stress distribution on the bar. Furthermore as the method is tedious to use it does not commend itself as a design procedure.

Ramakrishnan and Vijarangan in 1963 published the results of a series of tests on beams without web reinforcement. They concluded that the torsional strength of such beams could be calculated by ignoring the longitudinal reinforcement and by using an elastic distribution of stress and a maximum tensile stress criterion of failure. They proposed the following empirical relationship for the tensile strength

$$f'_t = 2.6\ C_u^{2/3} \qquad \ldots \qquad (2.6)$$

They further concluded that the addition of bending moment did not effect the torsional strength. As the stress criterion was based on their own tests in which only one concrete mix was employed correlation was good.

It is of interest to compare the predictions of the above theories with experimental results. For this purpose, Table 2.1 was prepared. In this table the test results are compared with the predicted ultimate torsion capacities obtained from the three theories. The table shows that although each investigator obtained good correlation between his own test results and theory, comparison with other results is not so satisfactory. The method of Gesund and Boston gives the best results, but even then the correlation is only fair. The theories of both Nylander and Ramakrishnan and Vijarangan are unreliable and at times unsafe. It may be concluded that none of the above theories offers a satisfactory basis for design.

2.1 (c) Shear and Torsion in Beams without Web Reinforcement

Despite the relative importance of the problem of shear force combined with torsion and bending, very little experimental work has been carried out for beams with longitudinal reinforcement alone. Nylander tested one series of beams and concluded that failure would occur when the sum of the shear stresses due to torsion and direct shear equals the tensile strength of the concrete. This approach would seem an oversimplification, as even in pure shear the stress at failure is not constant.

Based on Nylander's results Kemp suggested the following failure criterion:-

$$\frac{T}{T_o} + \frac{V}{V_o} = 1 \qquad \ldots \qquad (2.7)$$

Where T and V are failure loads, T_o is the failure torque in pure torsion, and V_o is the failure shear in flexure and torsion.

TABLE 2.1

A COMPARISON OF EXPERIMENTAL RESULTS WITH THE THEORIES OF NYLANDER, RAMAKRISHNAN AND VIJARANGAN AND GESUND AND BOSTON

Investigator	Beam No.	Failure Loads		$T_{exp}/T_{theor.}$		
		Torque kip. in.	Moment kip. in.	Nylander	Rama-krishnan and Vija-rangan	Gesund and Boston
Nylander	1	39.0	52.1	1.12	0.72	1.09
	2	31.2	52.1	0.97	0.72	0.88
	3	39.0	58.0	1.15	0.79	1.10
	4	35.1	58.0	1.07	0.79	0.99
	5	54.6	75.7	0.85	0.86	1.70
	6	50.7	75.7	0.90	0.80	1.61
	7	50.7	110.0	0.98	0.74	1.51
	8	54.6	110.0	1.03	0.80	1.63
	9	31.2	58.0	1.07	0.89	0.89
	10	19.5	58.0	0.89	0.89	0.89
Ramakrishnan and Vijarangan	B4	17.1	99.0	1.36	1.34	1.34
	B5	24.8	45.4	0.90	1.03	0.81
	B6	10.7	108.0	1.39	1.39	1.39
	C3	21.7	111.0	1.12	1.06	1.08
	C4	20.1	90.7	1.00	0.95	0.91
	C5	23.2	105.0	1.03	1.01	0.99
Gesund and Boston	3	58	58	1.10	0.69	1.09
	4	64	64	0.82	0.77	1.60
	5	43	86	0.74	0.83	1.48
	6	36	108	0.74	0.68	1.24
	7	59	177	1.05	0.93	1.31
	8	49	195	1.06	1.02	1.09
	9	42	83	0.44	0.49	1.50
	10	39	156	0.63	0.63	1.56
Summary			Maximum	1.39	1.39	1.70
			Minimum	0.44	0.49	0.81
			Mean	0.98	0.87	1.23
			Standard Deviation	21 %	24 %	24 %

Kemp suggested that this formula would be generally applicable, if V_o was based on a limiting stress of $2\sqrt{f'_c}$ and T_o based on $4\sqrt{f'_c}$.

2.2 BEAMS WITH WEB REINFORCEMENT

A large number of experimental investigations have been undertaken on the effects of combined longitudinal and transverse steel on the torsional strength of concrete beams. In some of this work attention has been focussed solely on the problem of pure torsion, but in a number of cases the problem of combined torsion, shear and bending has been investigated.

2.2 (a) Beams Subjected to Pure Torsion

For beams subjected to pure torsion there is agreement between most investigators that, irrespective of the amount or disposition of reinforcement, tensile cracks appear on the face of the beams with an inclination to the longitudinal axis of the beam of approximately 45^o, when the twisting moment is approximately the same as the cracking torque of a similar beam of plain concrete. Once the member has cracked the torsional stiffness is reduced. The behaviour beyond this stage depends primarily on the amount and position of the reinforcement.

Some disagreement exists about the final nature of failure. Particular points of disagreement are the contribution of the uncracked concrete to the torsional capacity, and whether yielding of the reinforcement actually causes failure.

The earliest researchers merely noted that reinforcement increased the torsional capacity. Rausch, (2.8) in 1929, attempted a rational solution to the problem of estimating the torsional capacity of reinforced concrete beams. Rausch considered the reinforced member as being analogous to a space frame in which tensile forces could be resisted by the steel. In this way he developed the following expressions for the required web and longitudinal steel:-

$$A_w = \frac{s}{2\ f_w\ b'\ d'}\ T_w \qquad \ldots \qquad (2.8)$$

and

$$A_L = \frac{b' + d'}{f_L\ b'\ d'}\ T_w \qquad \ldots \qquad (2.9)$$

In these expressions f_w and f_L represent the permissable stresses in the transverse and longitudinal steel. His expressions could be regarded as relationships between the area of reinforcement and the ultimate torsional strength i.e.

$$T = \frac{2\ b'd'}{s}\ f_w\ A_w \qquad \ldots \qquad (2.10)$$

if

$$A_L = \frac{2\ (b' + d')}{2}\ \frac{A_w}{s}\ \frac{f_w}{f_L} \qquad \ldots \qquad (2.11)$$

Rausch in discussing certain published test results stated that the very high twisting of members led him to believe that the reinforcement yielded before failure of the member. It is to be noted that Rausch did not allow for any contribution due to concrete shearing stresses to the torsional capacity. In Figure 2.1 the above equations are compared with the test results. Only within a certain range of values of $\frac{b'd'f_wA_w}{s\ T_c}$ (i.e. 0.5 to 1.0) is tolerable agreement with experimental results obtained. For lightly reinforced beams the failure torque is primarily influenced by the concrete and for heavily reinforced beams failure may occur

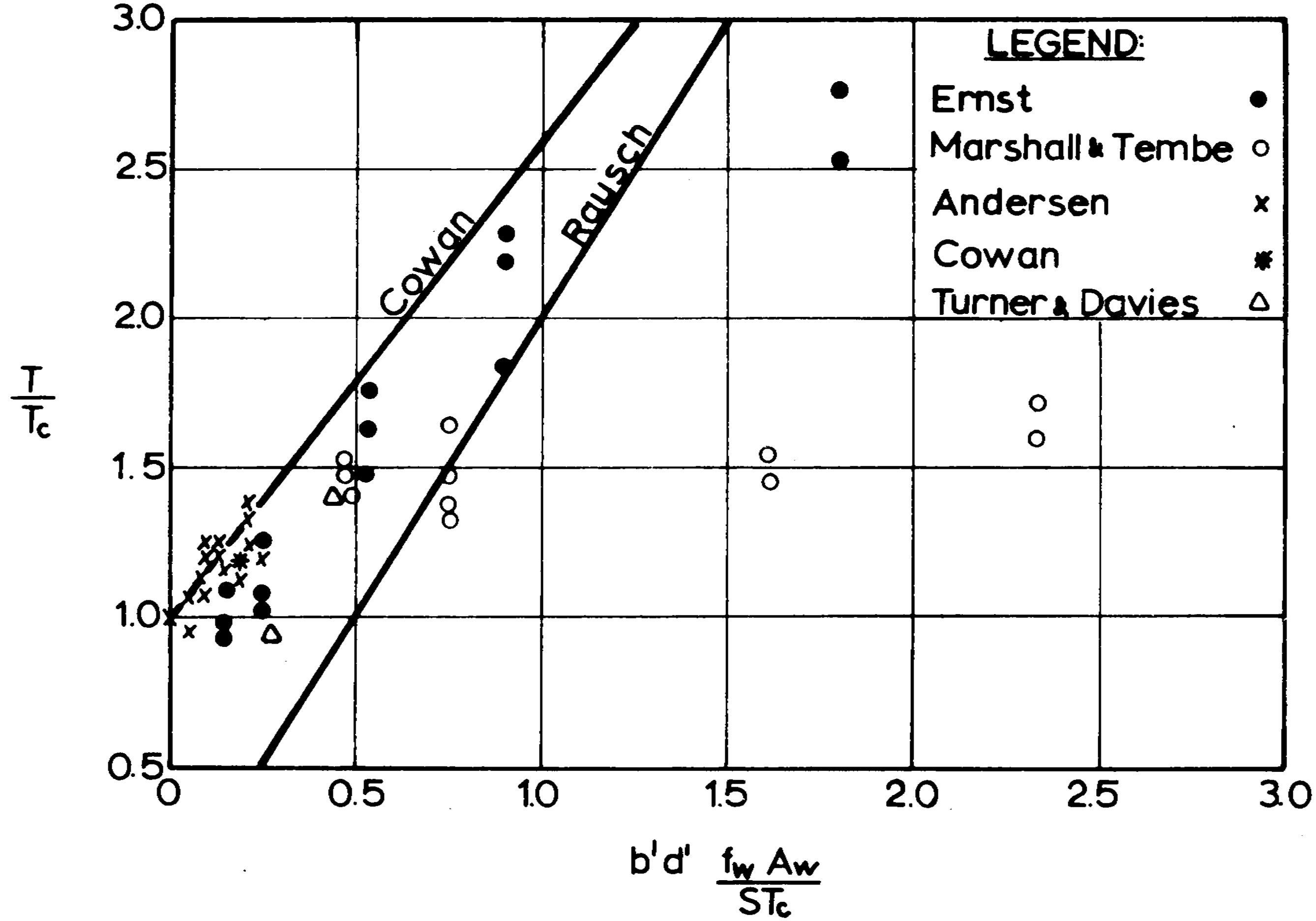


FIG. 2.1. TORSIONAL STRENGTH — AMOUNT OF TRANSVERSE REINFORCEMENT.

prior to yielding of the reinforcement. Rausch's expressions are still quoted in many modern codes for reinforced concrete construction: Germany, Egypt, Hungary and Poland (2.9).

Turner and Davies (2.10) in 1934 proposed a relationship for calculating the ultimate torsional capacity of a reinforced member.

$$T = T_c (1 + 0.25 p') \quad \ldots \quad (2.12)$$

where p' is the total percentage of reinforcement. The percentage of longitudinal steel is p'/2 and an equal volume per unit length of transverse steel is required. They recommended that p' should not be less than 1% if the section is to carry considerable torque.

Marshall and Tembe (2.11) in 1941 agreed with Turner and Davies proposal for values of p' less than 1.5, but recommended that the following expression be used for higher values

$$T = T_c (1.33 + 0.1 p') \quad \ldots \quad (2.13)$$

These criteria agree only approximately with the results shown in Figure 2.2. In plotting this figure values of the cracking torque have been obtained from companion test specimens without web reinforcement. If this method is used for design the cracking torque would have to be computed resulting in further loss of accuracy of the approach.

Andersen (2.12) carried out a large number of tests in 1935 and 1937. He employed spiral reinforcement in a large proportion of his reinforced specimens. Andersen analysed circular specimens by assuming a stress distribution and a failure criterion.

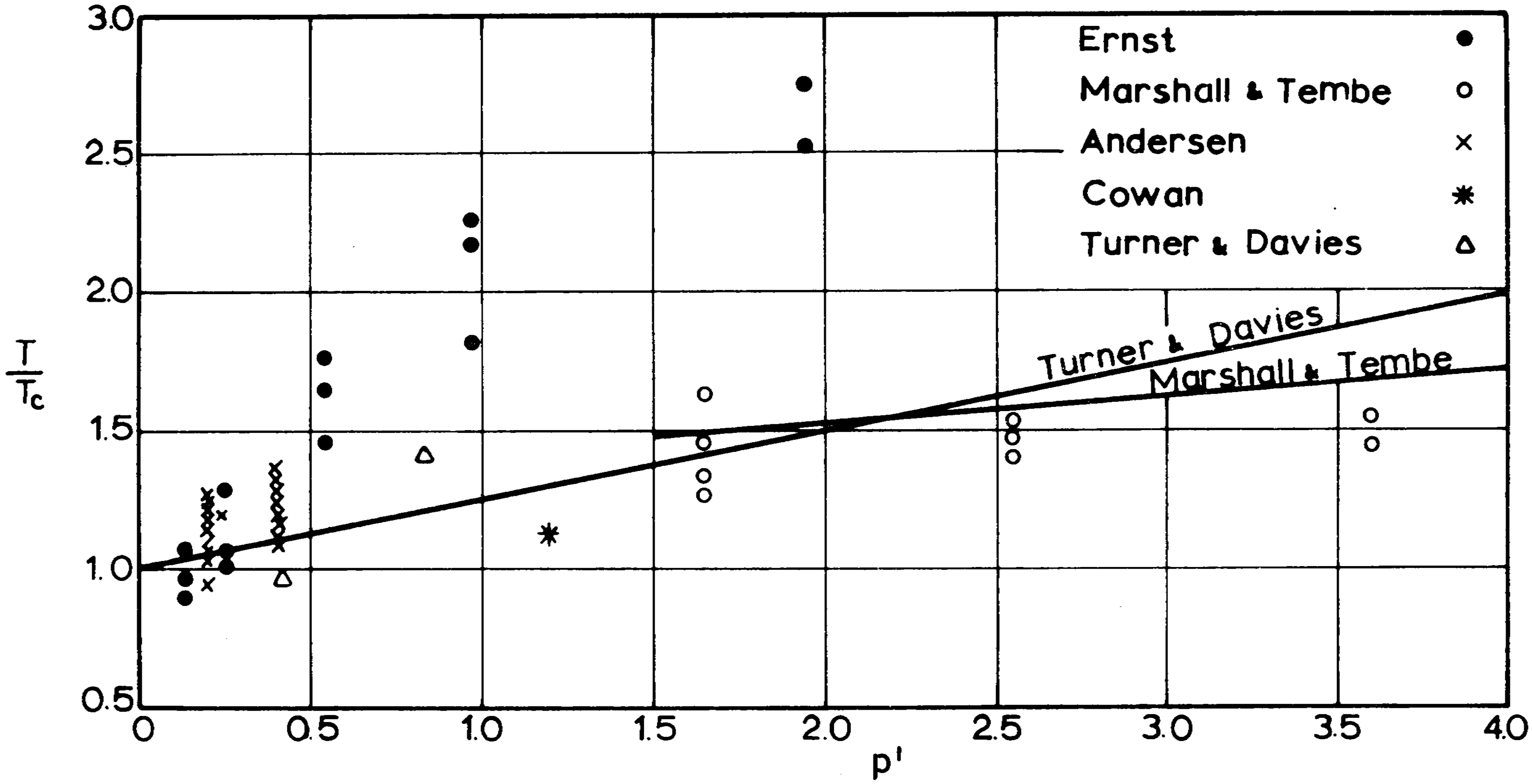


FIG. 2.2. ULTIMATE TORQUE / CRACKING TORQUE — PERCENTAGE OF REINFORCEMENT

The results were then related to square sections by "efficiency factors". The method obtained was neither simple nor accurate. Marshall and Tembe concluded that Andersen's formulae did not agree with test results.

Cowan (2.13), who has written extensively on the subject of torsion in concrete, proposed a theory for a working stress approach to torsion design. He considered that the torsional resistance of a concrete beam was provided partly by the concrete and partly by the steel. With several assumptions, he obtained from the theory of elasticity an equation for the contribution of the steel;

$$T_s = \frac{1.6\ b'd'}{s} A_w f_w \qquad \ldots \quad (2.14)$$

$$A_L = \frac{2(b' + d')}{s} A_w \qquad \ldots \quad (2.15)$$

The elastic formula is recommended by Cowan for the computation of the contribution of the concrete. Experimental results show that the effect of the steel on the ultimate strength of a web reinforced beam as compared with an unreinforced specimen is far less than the above equation would imply (See Figure 2.1). Additionally, the effect of reinforcement on the cracking torque has been observed by many investigators to be negligible. If, however, the above formula is used in conjunction with a low estimate of the contribution of the concrete, useable results may be obtained. Cowan's theory has been used in this manner in the Australian code (S.A.A. - CA 2).

All of the aforementioned proposals agree in one respect in requiring equal volumes of steel per unit length of beam in both the longitudinal and transverse directions.

Ernst (2.14) in 1957 conducted a series of tests to investigate the effect on torsional strength of variations in the ratio of longitudinal to transverse reinforcement. He found that both the longitudinal and transverse steel yielded at failure for wide variations in this ratio. Further he found that increasing the amount of longitudinal steel while maintaining a constant amount of transverse steel increased the torsional capacity of the beams. The theories outlined above cannot account for this increase in strength.

Lessig's ultimate equilibrium method does account for this increase. However, pure torsion is treated by Lessig as a particular case of the more general loading condition: bending, torsion and shear. In view of this fact a review of her work will be given in a later section of this report.

2.2 (b) Beams Subjected to Combined Torsion, Bending and Shear

Beams subjected simultaneously to the combined action of torsion, shear force and bending moment are of much more interest, from the practical viewpoint, than members with torsion only. This subject has received considerable attention of late years.

In 1953 Cowan (2.15) put forward a theory of strength for combined bending and torsion. He emphasised that his theory was concerned only with the visco-elastic limit or the point where pronounced cracking occurs. This point also corresponds with a marked change in the slope of load rotation curves. He divides the complete range of bending and torsion into "cleavage" failures associated with high torsion and "crushing failures" when bending moment is predominant. For cleavage failures he computed the visco-elastic limits from the uncracked stress distributions and a tensile failure criterion. This approach results in a predicted increase in torsional capacity with moderate bending. Although this effect was observed for Cowan's test beams, later work has shown that this effect is not universally true (Chapter 6).

Cowan concluded that "It is therefore reasonable to design reinforced concrete section subject to combined bending and torsion without reduction or adjustment to the maximum permissable concrete or steel stresses".

In light of more recent investigations the above statement seems to require modification.

An intensive study of the behaviour of reinforced concrete beams subject to combined torsion, bending and shear has been carried out in the Laboratory of Reinforced Concrete Structures - Moscow. The first major series of tests were conducted by N.N. Lessig (2.16). She concluded from this work that failure in most cases was initiated by yielding of the reinforcement. Other types of failure were observed but these appeared to be of less frequent occurrence. They will be discussed later. Lessig described two principal modes of failure.

In the first mode, which occurs most frequently with beams subjected to bending and torsion with negligible shear force (see Figure 2.3) cracks form on the sides and in the lower portions of the beam. The opening of these cracks is inhibited until either the longitudinal or both transverse and longitudinal reinforcement yields. As the steel yields the two lengths of beam rotate about an axis near the top face of the beam until the concrete in this face finally crushes.

A second failure mode (see Figure 2.4) may occur when relatively large shear forces are present in conjunction with bending moment and twisting moment. In such cases the inclined tension cracks are predominant on the side of the beam, where tensions arising from twisting moment and from direct shear force are additive. After yielding of the steel the lengths of beam rotate about an inclined hinge located near the face opposite to that in which the tension cracks first appeared.

Lessig derives expressions for predicting the failure loads for the two modes of failure described above. For the purpose of the analysis she assumes that the intersection of the failure surface with the beams faces are straight lines and further, that the inclination of these lines on the three sides corresponding to the tension cracks is constant. The assumption is also made that all steel traversed by the failure cracks yields.

In the analysis for Mode 1 the forces in the vertical legs of the hoops intersected by the failure crack, dowel forces and tensile stresses in the intact concrete are ignored. An inclined failure hinge of undetermined length q (see Figure 2.3) is assumed. Moments about this hinge are calculated for the tensile forces in the longitudinal reinforcement and in the bottom horizontal parts of the transverse steel. These moments are then equated to the components of the external moments about this axis. The critical length of the failure hinge which makes the moment a minimum is then determined. The depth of the compression zone is then found by equating the compressive forces perpendicular to the failure surface to the components of the tensile forces in the steel.

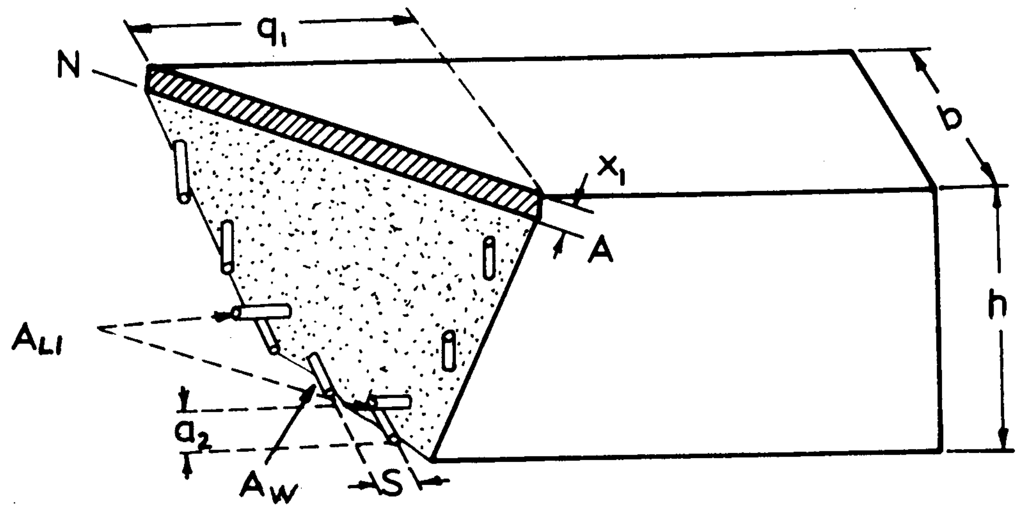


FIG. 2.3. FAILURE SURFACE MODE 1.

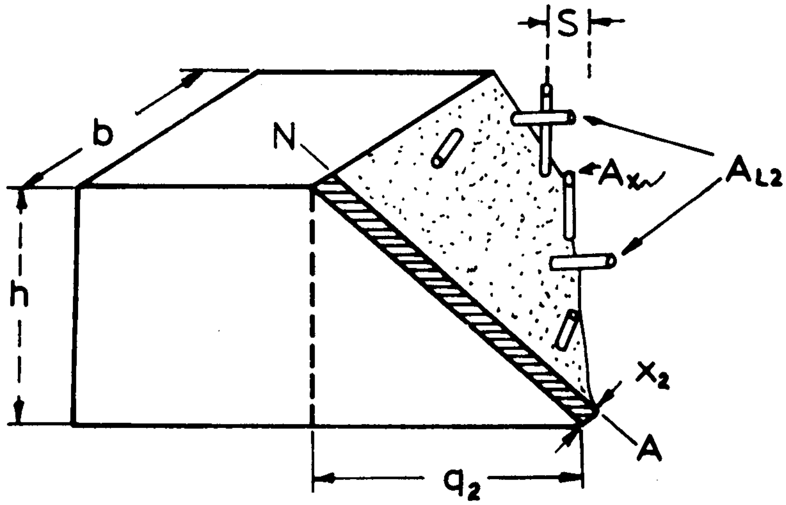


FIG. 2.4. FAILURE SURFACE MODE 2.

The depth of compression computed on this basis is usually quite small. It is difficult to see how this shallow compression zone can resist the longitudinal forces required for equilibrium with the tensile forces in the main longitudinal steel. Indeed it would appear that this step in Lessig's analysis is a cause of some inaccuracies in the final equations.

The analysis equations were presented in various forms in papers by Lessig and subsequent Russian investigators. The following expressions are in the form given in the more recent papers and in the Russian code.

$$T_1 = \frac{(h - a_2 - x_1/2)}{(\frac{q_1}{b} + \frac{1}{\psi})} \left(A_{L1} f_{L1} + \frac{A_w f_w q_1^2}{s(2h + b)} \right) \quad \ldots \quad (2.18)$$

$$q_1 = \frac{b}{\psi} + \sqrt{(\frac{b}{\psi})^2 + \frac{A_{L1} f_{L1} \; s\; (2h + b)}{0.8 A_w f_w}} \quad \ldots \quad (2.19)$$

but $q_1 \nmid 2h + b$

and

$$x_1 = \frac{b}{0.85 \; f'_c \; (q_1^2 + b^2)} \left(A_{L1} f_{L1} + \frac{A_w f_w q_1^2}{s(2h + b)} \right) \quad \ldots \quad (2.20)$$

By employing similar methods the following expressions are obtained for Mode 2.

$$T_2 = \frac{(b - a_1 - x_2/2}{q_2(1 + \delta \;)} h \left(A_{L2} f_{L2} + \frac{A_w f_w q_2^2}{s(2b + h)} \right) \quad \ldots \quad (2.21)$$

$$q_2 = \sqrt{\frac{A_L f_{L2} \; s(2b + h)}{A_w f_w}}$$

but $q_2 \nmid 2b + h$,

and

$$x_2 = \frac{h}{0.85\ f'_c\ (q_2^{\ 2} + h^2)} \left(A_{L2} f_{L2} + \frac{A_w f_w q_2^{\ 2}}{s(2b + h)} \right) \qquad \ldots \qquad (2.22)$$

The predicted failure torque of the member is then taken as the lesser of the two values T_1 and T_2.

The complete derivations are given in Lessig's work (2.17).

Lessig's test and a series of tests by Lyalin (2.18) have shown that for cases where yield of steel occurs at failure the theory presented by Lessig is reasonably accurate.

Chinenkov (2.19) who conducted a series of tests on reinforced concrete beams subject to combined actions concluded that Lessig's formulae predict failure loads agreeing substantially with test values. Experimental values slightly higher than the predicted values could be accounted for by the concrete tensions ignored in the theory. In connection with Lessig's assumption regarding the constant inclination of the tension crack Chinenkov stated that this assumption did not agree with the observed facts.

It is to be noted that the formulae given above have been based on the assumption that both longitudinal and transverse steel yield. This condition imposes several limitations on the amount and distribution of the reinforcement which may be used.

In the first instance the amount of reinforcement must be limited so that the concrete in the compression zone will not crush before the steel yields.

Lessig conducted a series of tests in 1957-58 to attempt to define this limit empirically.

She observed that even with high percentages of steel this mode of failure rarely occurred for values of ψ in excess of 0.2. From those tests in which compression failure occurred she obtained the following relation between the depth of the compression zone and :

$$\frac{X_c}{h - a_2} = 0.55 - 0.7\sqrt{\psi} \quad (0 < \psi < 0.2) \qquad \ldots \qquad (2.23)$$

For cases where X_c, from the above, was greater than or equal to $2a_2$ the relation between steel areas and X_c would be

$$(A_{L1}f_{L1} - A_{L3}f_{L3}) < 0.85\ f'_c\ b\ (1 + 5\psi)X_c \qquad \ldots \qquad (2.24)$$

whereas if the indicated value of X_c were less than $2a_2$ the steel area would then be given by

$$A_{L1}f_{L1} \leq 0.85\ f'_c\ b\ (1 + 5\psi)\ X_c \qquad \ldots \qquad (2.25)$$

The empirical equation (2.23) was based on a rather small number of tests, Lessig suggested that there is need for further work on this aspect of the problem.

A much more common case of concrete compression failure occurs when the ratio of twisting moment to bending moment is higher than for the case mentioned above.

A number of beams tested by Lessig failed in this manner. She assumed that the ultimate twisting moment in such a case could be expressed by a relation of the form

$$T = K\ b^2\ h'\ f'_c \qquad \ldots \qquad (2.26)$$

As K would appear dependent upon the relative values of twisting moment and direct shear force, she examined the effect of a ratio involving these two actions. For the type of failure being considered she concluded that there was no correlation between this ratio and the value of K. As K varies between 0.07 and 0.12 she suggested that a failure would not occur if

$$\frac{T}{b^2\ h\ f'_c} \leq 0.07 \qquad \text{...............(2.27)}$$

To test the effectiveness of this limitation, i.e. its ability to exclude unconservative results, Figure 2.5 has been prepared. In this figure the ratio T_{exp}/T_{theor} has been plotted against the value of $T_{theor}/b^2\ h\ f'_c$. The general trend of the results shown in this figure suggest that the criterion is satisfactory.

Various investigators have noted that complete stress redistribution can take place between transverse and longitudinal steel - allowing both to yield point - for wide range of ratios of these steel quantities.

Lessig and Lyalin attempted to establish empirically the limits of the ratio of transverse to longitudinal steel for which yielding of both steels could be guaranteed.

They fixed limits as follows;
for mode 1,

$$0.5 \leqslant \frac{A_w f_w b}{A_{L1} f_{L1}\ s} \left(1 + \frac{2}{\psi} \sqrt{\frac{b}{2h + b}}\right) \leqslant 1.5 \qquad \text{...} \qquad (2.28)$$

and for mode 2,

$$0.5 \leqslant \frac{A_w f_w\ h}{A_{L1} f_{L1}\ s} \leqslant 1.5 \qquad \text{...} \qquad (2.29)$$

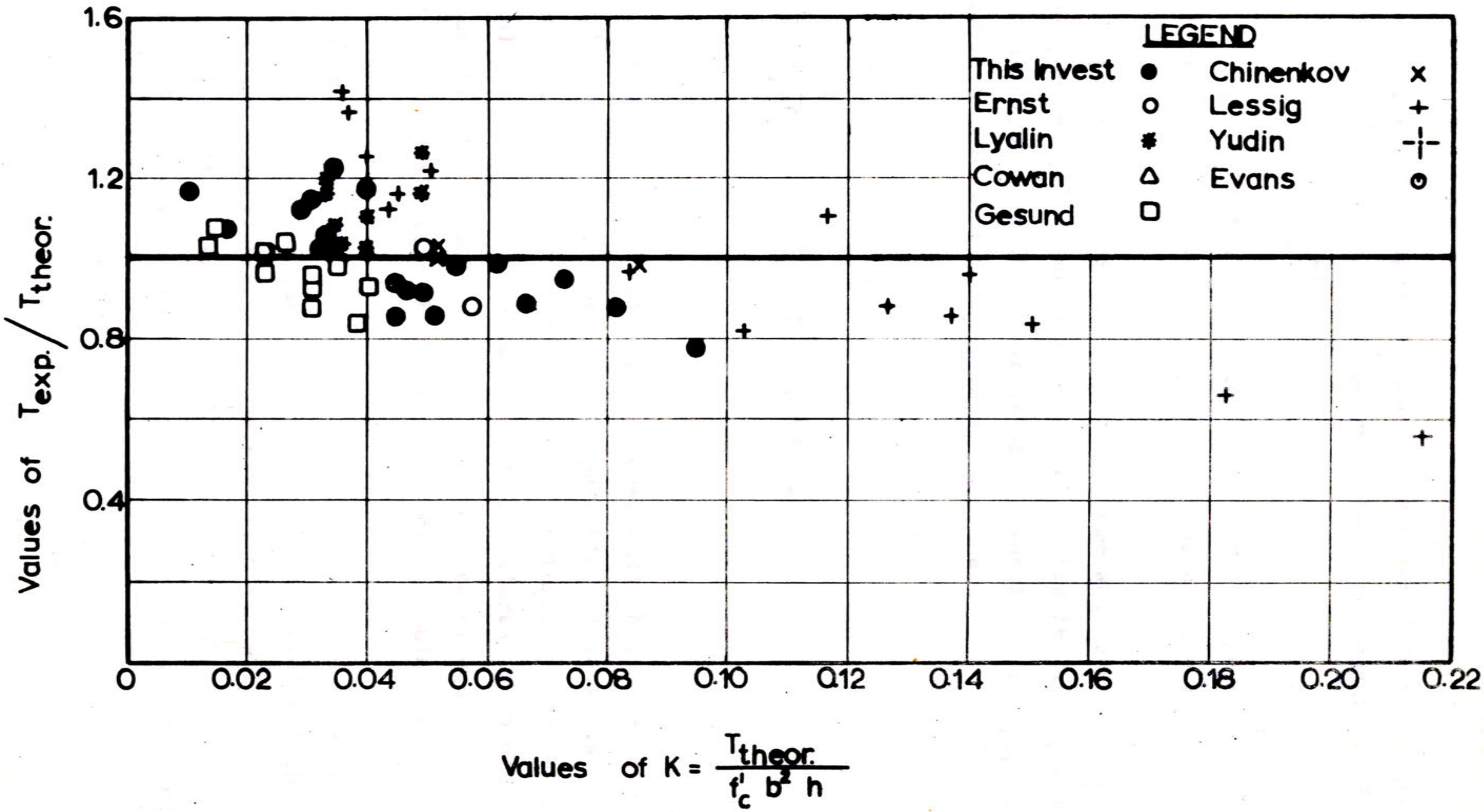


FIG 2.5 ACCURACY OF LESSIG'S THEORY VERSUS K

Lessig further suggested that beams which do not satisfy these restrictions could be analysed by ignoring the excess quantity of transverse or longitudinal steel.

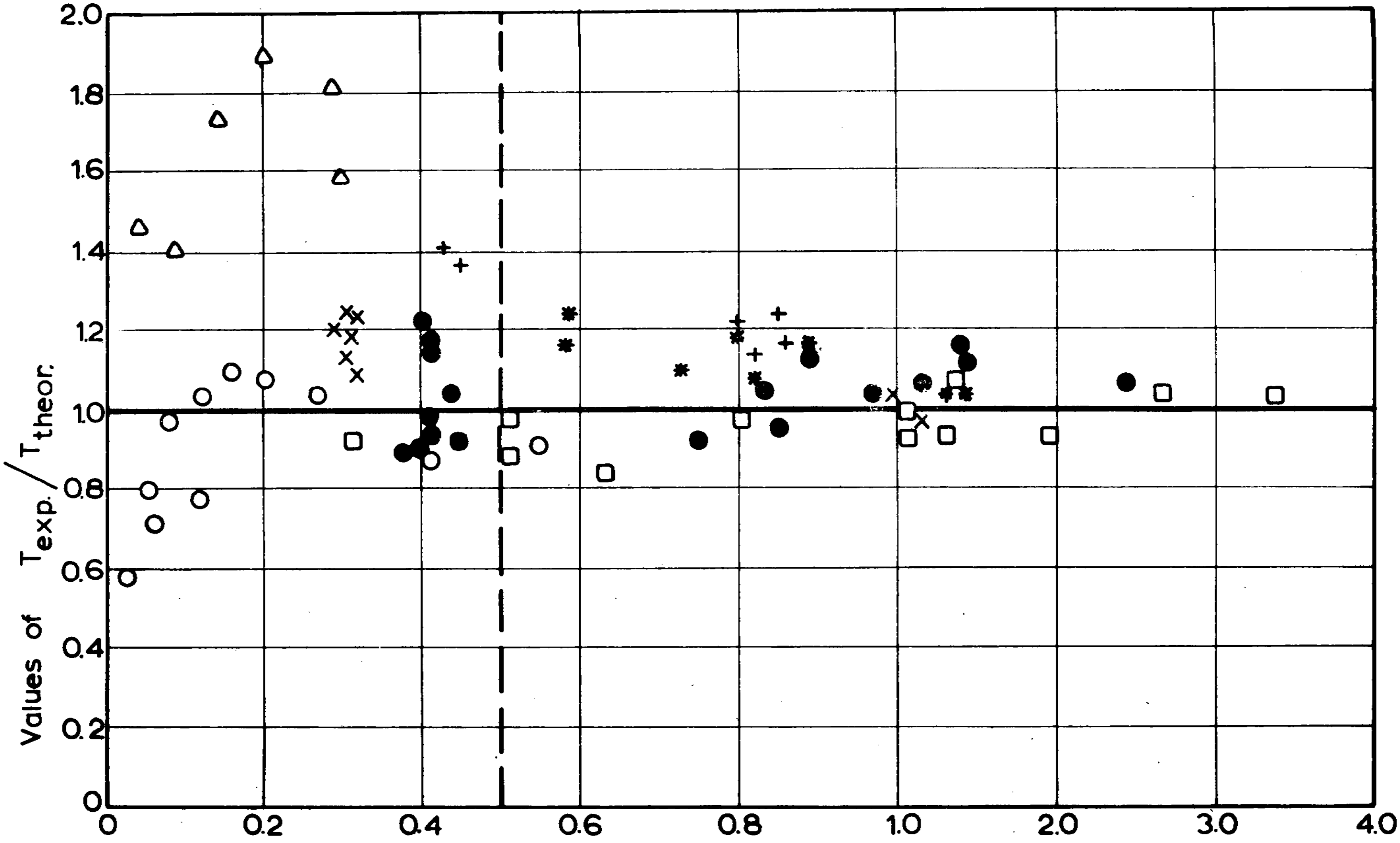
In all of the tests in which these limits were based the transverse steel was spot-welded to the longitudinal steel.

These restrictions are both cumbersome and severe. To investigate the necessity for such restrictions Figure 2.6 has been prepared. In Figure 2.6 the accuracy of the theory (i.e. T_{exp}/T_{theor}) is compared with values of the parameter, R_1, where

$$R_1 = \frac{A_w f_w}{A_{L1} f_{L1}} \quad \frac{b}{s} \quad \left(1 + \frac{2}{\psi} \quad \sqrt{\frac{b}{2h + b}}\right)$$

An examination of this figure shows that while the theory gives less accurate results for values of this parameter of less than 0.5, this restriction could safely be liberalized. Unfortunately, not enough experimental results are available for mode 2 failures to enable a similar check to be made on the restriction emposed by equation (2.29). However it would seem likely that this restriction could also be relaxed.

The effect of the variation of the bending and twisting moments within the failure zone was made the subject of a theoretical investigation by Lessig. Amended design equations taking such variations into account are extremely complex. Considering that the theory ignores factors such as tension in the concrete etc., this refinement seems unwarranted.



Values of $R_1 = \dfrac{A_w f_w}{A_{L1}\ f_{L1}} \cdot \dfrac{b}{S}\left[1 + \dfrac{2}{\psi}\sqrt{\dfrac{b}{2h+b}}\right]$

FIG 2.6 ACCURACY OF LESSIG'S THEORY VERSUS R_1

The mode 2 failure which has been described earlier occurs in the presence of moderate shear force with yielding of the steel. Lessig found that a more normal failure occured under shear force in combination with torsion and bending. Based on an empirical analysis of test results Lessig gave the following equation for the strength of a member failing in this manner,

$$V = \frac{V_o}{1+1.5/\delta} \qquad \ldots\ldots\ldots \ (2.30)$$

where

$$\delta = \frac{Vb}{2T}$$

and V_o is the shear capacity of the beam in the absence of torsion, given in the Russian code by

$$V_o = \sqrt{(0.51\ f'_c\ b\ (h-a_2)^2 \quad \frac{A_w f_w}{s} \quad n)} - A_w f_w n \qquad \ldots\ldots \ (2.31)$$

where n - number of "legs" of the stirrups.

Chinenkov performed two series of tests on reinforced concrete beams subject to combined bending and torsion. In one series he examined the behaviour of beams subject to combined bending and torsion. In one series he examined the behaviour of beams in which the web reinforcement consisted of only vertical bars near the side faces. For such beams he concluded that the maximum torque is given by

$$T = \frac{b^2}{2} \; (h - b/3) \; f'_t \qquad \qquad ....(2.32)$$

In the other series, the beams were reinforced with hoops and longitudinal steel, the main aim being to check the theories proposed by Lessig. He found that her theories gave results in substantial agreement with experiments although on the conservative side.

To evaluate the accuracy of Lessig's theory and the ultimate strength theories proposed by other investigators, an analysis of all the available test data has been carried out. The results of this work are presented in Table 2.2. To eliminate abnormally reinforced sections, only those beams which satisfy the restrictions set out in equations (2.25) and (2.27), were considered. It can be seen from this table that the theory of Lessigs is only moderately accurate. The mean of the values of T_{exp}/T_{theor} for the tests is 1.62 with a standard deviation of 48%.

In analysing this test data the excess quantity of longitudinal or transverse steel outside the limits set out in equations (2.28 and 2.29) was ignored in accordance with Lessig's recommendation. If attention is further restricted to beams satisfying these limits and the beam failing in Modes 1 or 2, the theory can only be applied to 18 tests out of the total of 151 available test results. For this group, marked with an asterisk in Table 2.2, T_{exp}/T_{theor} has a mean of 0.98 $\pm$ 14%.

It should be noted that to determine the strength of a section and to check all the limitations quite a considerable amount of calculation is involved. Furthermore the method is an analysis approach and design must be carried by assuming all dimensions and reinforcement.

TABLE 2.2

A COMPARISON OF EXPERIMENTAL RESULTS AND THE PUBLISHED THEORIES FOR WEB REINFORCED BEAMS

Part 1. Pure Torsion

Investigator	Beam	Torque kip. in.	T_{exp}/T_{theor} Lessig	Yudin	Evans
Ernst	3TR3	34.3	2.18	2.01	0.31
	3TR1	49.7	1.58	1.45	0.52
	3TR15	61.7	1.12	1.03	0.82
	3TR30	76.0	0.82*	1.04	2.98
	4TR3	35.0	2.24	2.05	0.23
	4TR7	54.8	1.76	1.60	0.40
	4TR15	74.0	1.36	1.24	0.63
	4TR30	85.0	0.79*	0.89	1.27
	5TR3	43.0	2.75	2.52	0.15
	5TR7	59.7	1.91	1.75	0.22
	5TR15	76.5	1.40	1.28	0.30
	5TR30	92.6	0.85	0.77	0.46
Cowan	R3	71.8	4.05	3.30	0.33
	Mean		1.49	1.62	0.66
	Standard Deviation		40%	40%	80%
	Number of Tests		13	13	13

Part 2. Bending and Torsion

Investigator	Beam	Torque kip.in.	Moment kip.in.	T_{exp}/T_{theor} Lessig	Yudin	Evans	Gesund
Cowan and Armstrong	R5	75.4	75.4	3.86	3.47	0.67	0.88
	R2	79.0	158.0	3.84	3.64	1.04	1.27
	R1	43.0	258.0	1.95	1.98	1.32	1.51
	S1	82.6	206.5	2.95	2.85	1.26	1.52
	S4	64.6	258.4	2.23	2.23	1.41	1.63
Gesund and Boston	1	79.0	79.0	0.90*	1.07	0.79	1.02
	2	102.0	102.0	0.91	1.26	0.93	1.32
	3	61.0	122.0	0.93*	1.09	0.90	1.13
	4	67.0	134.0	0.95	1.20	0.89	1.25
	5	49.0	147.0	0.97*	1.16	0.97	1.19
	6	56.0	168.0	1.10	1.33	1.00	1.37
	7	43.0	173.0	1.07	1.25	1.07	1.29
	8	44.0	176.0	1.08	1.27	0.97	1.31
	9	60.0	120.0	1.00	1.46	0.64	1.04
	10	44.0	176.0	0.84*	1.07	0.77	1.21
	11	68.0	138.0	0.78*	1.12	0.76	1.18
	12	53.0	213.0	0.90*	1.26	0.94	1.46
Chinenkov	B28 0.1	48.6	486.0	1.04*	1.14	1.08	1.23
	B28 0.1a	46.9	469.0	1.00	1.11	1.05	1.20
	B28 0.4	146.0	365.0	1.40	1.99	0.94	1.49
	B28 0.4a	139.0	347.0	1.33	1.90	0.90	1.42
	B28 0.4b	146.0	365.0	1.40	1.99	0.93	1.25
	B28 0.4c	153.0	382.0	1.44	2.12	0.89	1.36
	B28 0.4d	125.0	313.0	1.19	1.67	0.85	1.49
	B28 0.4e	132.0	330.0	1.28	1.77	0.86	1.54
Mean				1.51	1.70	0.95	1.29
Standard Deviation				56%	40%	19%	15%
Number of Tests				25	25	25	25

Part 3. Bending, Torsion and Shear

Investigator	Beam	Torque kip.in.	Moment kip.in.	Shear kips	T_{exp}/T_{theor} Lessig	Yudin
Lessig	BIII2	146.0	243.0	9.24	1.51	1.21
	BIII2A	151.0	261.0	10.21	1.54	1.31
	BIII5	156.0	416.0	15.56	1.56	1.38
	BIII5A	151.0	416.0	15.52	1.47	1.36
	BIII6	92.0	156.0	4.07	1.14*	1.62
	BIII6A	83.4	156.0	4.16	1.11*	1.56
	BIII7A	90.4	313.0	8.06	1.20*	1.55
	BIII7	83.4	278.0	7.16	1.09*	1.41
	BIII8	114.5	191.0	4.99	1.87	2.07
	BIII8A	111.0	191.0	4.99	1.80	1.85
	BIII9	78.0	313.0	4.63	1.33	1.45
	BIII9A	79.0	313.0	4.99	1.80	1.85
	BIII9	78.0	313.0	4.63	1.33	1.45
	BIII9A	79.0	313.0	4.71	1.39	1.55
	WB	53.0	132.0	6.64	2.73	3.35
	WBA	57.3	143.0	7.16	2.85	3.45
Lyalin	B8 0.1	52.0	520.0	12.52	1.04	1.18
	B8 0.1A	55.5	555.0	13.36	1.04	1.17
	B7 0.2	93.8	468.0	11.30	1.69	1.28
	B2	139.0	694.0	16.65	1.14*	1.33
	B2A	139.0	694.0	16.65	1.06*	1.24
	B2A	139.0	694.0	16.65	1.06*	1.24
	B3	194.0	486.0	17.48	1.30	1.21
	B3A	194.0	486.0	17.48	1.30	1.21
	B5	194.0	972.0	23.24	1.07*	1.25
	B5A	194.0	972.0	23.24	1.16*	1.36
	B6	167.0	833.0	20.19	1.14	1.34
	B6A	181.0	903.0	21.88	1.24	1.46

Part 3 (contd.)

Investigator	Beam	Torque kip.in.	Moment kip.in.	Shear kips	T_{exp}/T_{theor} Lessig	T_{exp}/T_{theor} Yudin
Yudin	7	9.9	49.3	2.51	3.95	1.49
	18	7.9	39.4	2.00	2.77	1.59
	19	7.9	39.4	2.00	2.77	1.59
	21	7.9	39.4	2.00	2.77	1.59
	Mean				1.66	1.57
	Standard Deviation				44%	34%
	Number of Tests				29	

then checking the capacity, and if necessary modifying the section and repeating the process.

Yudin (2.20) observed that the mechanism employed by Lessig only satisfied two equations of equilibrium. This criticism was also mentioned by Hsu when discussing Gesund's work. To satisfy equilibrium completely it is necessary to consider shear and other forces in the compression zone. The existence of these forces render Lessig's formula for the depth of compression rather dubious. Yudin, however, ignored the distribution of stresses in the compression zone and his mode also fails to completely satisfy equilibrium. Instead of taking the moment of the internal forces about the axis of the failure hinge and equating to the components of the external moments about this axis, he contends that the moments of both internal and external actions should be taken about the longitudinal axis of the beam and an axix perpendicular to it. Using the same failure modes as Lessig and assuming that the failure crack always spirals around three sides of the beam at 45° to the beam axis, his method gives the following results.

For mode 1 the ultimate twisting moment is

$$T = \frac{A_w f_w}{s} b'd' \qquad \qquad \ldots\ldots\ldots(2.33)$$

(Note: This is the same as Rausch's formula).

Equating the moments of the internal forces about a transverse axis to the external bending moment, and noting that the forces in the vertical legs of the stirrups contribute to the internal moment, the following expressions are obtained,

$$f_{L1}A_{L1} = \left(\frac{1}{h-a_2-\frac{x_1}{2}}\right)\left(M + \frac{T(b'+d')}{2b'}\right) \qquad \ldots\ldots(2.34)$$

or $$A_{L1} = \frac{1}{f_{L1}} \left(\frac{M}{h-a_2-\frac{x_1}{2}}\right)(1+C_1\psi): \text{ where } C_1 = \frac{(b'+d')}{2b'} \qquad \ldots(2.35)$$

or $$M = \frac{M_u}{1+C_{1\psi}} \qquad \ldots\ldots\ldots\ldots\ (2.36)$$

To guard against a mode 2 failure the twisting moment T is replaced by an effective twisting moment T'.

$$T' = T(1 + \frac{Vb'}{2T}) \qquad \ldots\ldots\ldots\ldots(2.37)$$

and the corresponding area of hoop reinforcement is

$$A_w = \frac{T' \ s}{2f_w b'd'} \qquad \ldots\ldots\ldots\ldots\ (2.38)$$

Additional longitudinal steel in a side face is required to balance the moment about a vertical axis of the forces in the horizontal legs of the hoops. This quantity is

$$A_{L2} = \frac{T'(b' + d'}{2d'f_{L2}(b-a_2-\frac{x_2}{2})} \qquad \ldots\ldots\ldots\ldots\ (2.39)$$

Yudin in a subsequent paper (2.20) presented experimental vertification of his theory. A comparison of this theory with the results from Yudin's tests and other reported tests is given in Table 2.2. It can be seen that the method is conservative and moderately accurate. It should be noted that for design purposes Yudin's formulae are very much simpler than those of Lessig.

Gesund, Shuette, Buchanan and Gray (2.21) tested a number of reinforced beams subject to combined torsion and bending. In their analysis they have considered mode 1 failures only, although they mention the possibility of mode 2 failures. Their approach is essentially the same as that proposed by Yudin, in that they consider moments about longitudinal and transverse axes, however the inclinations of the failure cracks to the longitudinal axis are taken as 45^{o} on the sides and θ on the bottom. The value of θ is taken as $\text{Cot}^{-1}0.5$ for $\psi >$ 0.25 and 90^{o} for $\psi <$ 0.25.

The resulting expression for the maximum bending moment in the presence of torsion is

$$M = \left(\frac{M_u}{1 + C_2\psi}\right) \qquad \text{.... (2.40)}$$

where M_u is the ordinary ultimate moment for an under-reinforced beam and

$$C_2 = \frac{d'(h+b \cot \theta)}{b'd' + (h-a_2)b' \cot \theta} \qquad \text{..... (2.41)}$$

The torsional capacity of the member is considered to be the greater of the two moments; one based on a consideration of dowel forces and the other based on the premise that the hoop steel yields before failure.

The general form of the equation which predicts the torsional resistance by considering the effect of dowel forces is

$$T = F_c \left(r_c + \frac{1}{r_c} \sum \left(\frac{2(h-2a_4)}{s} r_t^2 + r_i^2 \right) \right) \qquad \ldots\ldots (2.16)$$

where r_t is the average distance from the failure hinge to the stirrups and the other terms have the meanings given earlier in the discussion of beams without transverse steel.

The general form of the other equation which predicts the torsional resistance and which is obtained by taking moments about the failure hinge of the forces in the vertical and bottom legs of the transverse steel - assumed to be yielding - is

$$T = \frac{A_w f_w}{s} \left((b - 2a_3) \quad (h - 2a_4) + (h - a_2) \cot \theta \right) \qquad \ldots\ldots (2.17)$$

where θ is the angle between the failure crack on the bottom of the beam and the beam axis.

For the case of pure torsion it might be expected that $\cot \theta$ would be approximately unity. With $\cot \theta = 1$ the above expression is essentially the same as Rausch's formula.

The theory of Gesund shows reasonable agreement with the reported test results. The results of an analysis of test data is shown in Table 2.2, where T_{exp}/T_{theor} has a mean of 1.29 $\pm$ 15%.

As trial and error calculations are required to apply Gesund's dowel force equations, the method is far too tedious for design. The method is restricted to bending and torsion and does not account for shear.

Further, no limitations have been placed on the application of the theory although it is certain that unconservative results would be obtained for over-reinforced beams.

Evans and Sarkar (2.22) in a recent publication developed another ultimate strength theory for bending and torsion. Again the basic mechanism of a spiral tension crack on three sides and a compression hinge near the fourth side is assumed. The shape of the tension spiral is computed from the direction of the principal tensile stresses prior to cracking and an empirical crack trajectory on the side of the beam. This approach would appear to be more accurate although more complex than other simpler assumptions. An attempt is made to compute the direction of the compression hinge from the uncracked principal compressive stresses. This direction is found to vary from normal to the longitudinal axis to an inclination of about 15^{o}. For the purpose of simplicity the direction is assumed to be a constant 45^{o}. Other theories and experimental observation indicate that although the inclination of the compression hinge may vary, for predominantly torsion it is inclined at a smaller angle to the longitudinal axis than 45^{o}. This assumption may lead to undue emphasis being placed on the contribution of the longitudinal reinforcement to the torsional resistance.

The comparison with experiments given in Table 2.2 shows that Evan's theory has serious faults. The overemphasis on the longitudinal steel leads to unconservative results for pure torsion and for some bending and torsion test results. It is suggested that the method should not be used in the design of structural members.

The ultimate strength theories proposed by Lessig, Yudin, Gesund et al., and Evans and Sarkar show many similarities. Each theory is based on a failure mechanism which consists of a spiral tensile crack on three sides and a compression hinge on the fourth side. The reinforcement crossed by the tensile spiral is assumed to yield at failure. At this stage each investigator makes various assumptions to enable the relationships between the internal and external moments to be developed. Lessig's method, with some modifications would seem to offer the most accurate estimate of the failure load of a given section. In particular the formula for the depth of compression (Eq. 2.20 and 2.22) would appear to be inaccurate and leads to the severe restrictions on the amounts of reinforcement that can be considered in analysis. The method proposed by Yudin gives reasonable accuracy and is quite suitable for designing a section when the loads are given.

It must however be concluded that none of the existing ultimate strength theories satisfy the dual criterion of accuracy and simplicity.

2.3 DEFORMATION OF BEAMS

A large amount of research has been carried out on the deflection of reinforced concrete beams under bending. A brief summary of some of this work will be given in this section, as bending constitutes a limiting case of bending and torsion. Much less work is available in the fields of deformations in pure torsion and bending and torsion.

2.3(a) Deflections of Beams in Simple Flexure.

It is convenient to discuss the deflections of a reinforced concrete beam in the same terms as used in the deflections of an elastic homogeneous beams. In this simpler case the curvature, ϕ, is given by,

$$\phi = \frac{M}{EI}$$

where E is the stiffness of the material and I is the moment of inertia of the section. The product EI is called the flexural rigidity. If the curvature is known along the beam the simple integration technique will lead to the deflection.

In most of the theories of deflection in reinforced concrete an attempt is made to evaluate an equivalent flexural rigidity, EI, under varying conditions of loading. The flexural rigidity of a concrete beam may be considered to vary between two limiting states; (i) the gross transformed section, and (ii) the cracked transformed section.

The gross transformed section corresponds to the initial, uncracked state of the beam. This value can be used to compute the deflection in the uncracked state. A simpler, though less exact, approach is to use the gross section without taking into account the reinforcement. In this case the section modulus, Ig, is given by,

$$Ig = \frac{bh^3}{12}$$

The use of the gross section to compute the deflection at service loads has been recommended by the Portland Cement Association (2.23); and the A.C.I. code (2.24). The A.C.I. code restricted this method to lightly reinforced beams.

The cracked transformed section represents a state where tensile cracking and bond slip are so extensive that the concrete in tension can be ignored. The use of the cracked trsnformed section has been advanced by Maney (2.25), Myrlea (2.26), Yu and Winter (Method A) (2.27), and A.C.I. code (for heavily reinforced beams), Blakey (2.28) and others. The value of the section moment of inertia Ic for the cracked section is:

$$Ic = A_L \, jd^2 \, (1-k).$$

The faults of these two simple methods are demonstrated by a comparison of their predictions with an actual load deflection curve in Figure 2.7. It can be seen that the use of Ig underestimates the deflection by not considering the effect of cracking. On the other hand the cracked stiffness, Ic, overestimates the deflection by ignoring the concrete between tensile cracks and the stiffer uncracked areas in low moment regions.

Many investigators have derived expressions which give values of the rigidity that lie somewhere between the above extremes. Empirical formulas have been proposed by Murashev (2.29), Yu and Winter (Method B), Branson (2.30) and C.E.B. (2.31). The rigidity of the section is given by by Branson as,

$$I = \left(\frac{M_c}{M}\right)^4 Ig + \left(1 - \left(\frac{M_c}{M}\right)^4\right) I_c.$$

where M_c is the cracking moment. The A.C.I. committee (2.32) found that methods of this type can be expected to give

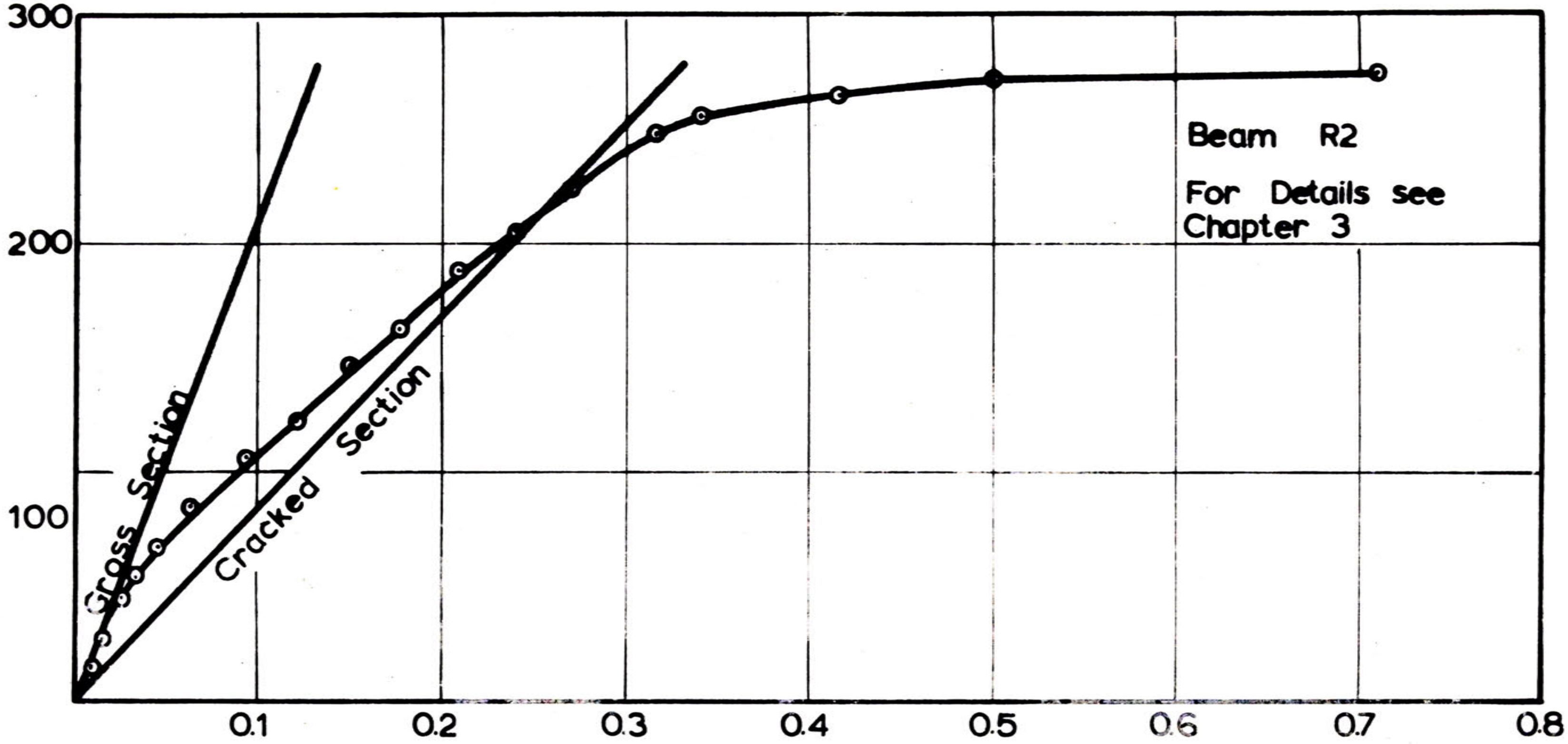


FIG. 2.7 COMPARISON OF SIMPLE DEFLECTION THEORIES WITH AN ACTUAL LOAD DEFLECTION CURVE.

predicted results within $\pm$ 20% of experimental deflection.

Brettle (2.33) devised a rational theory which allows for the stiffening effect of the tensile concrete between cracks. He assumed a linear bond stress distribution and from this assumption found the crack spacing and steel stresses. From the steel and concrete stresses the curvature and thence the deflection could be obtained. Although the method does provide a rational approach, it is extremely complex.

2.3 (b) Rotation of Beams Loaded in Pure Tension

Only limited information is available in the literature on the rotation of beams loaded in pure torsion.

Various investigators have discussed the elastic behaviour of specimens. It was found (2.1, 2.10, 2.11, 2.12) that up to the cracking torque, the torque-twist curves were very nearly linear. They further found that the amount of reinforcement had no apparent effect on the slope of the torque-twist line.

Only one detailed treatment of post cracking behaviour appears to have been made. In 1957 Ernst (2.14) reported the results of an investigation into web reinforcement specimens, in which considerable attention was given to their rotation behaviour.

Tests were carried out in which the percentage of longitudinal and web reinforcement were varied. One set of the torque-twist curves obtained is reproduced in Figure 2.9. Ernst described the effect of reinforcement on the ductility of the test beam but coult not draw any quantitative conclusions.

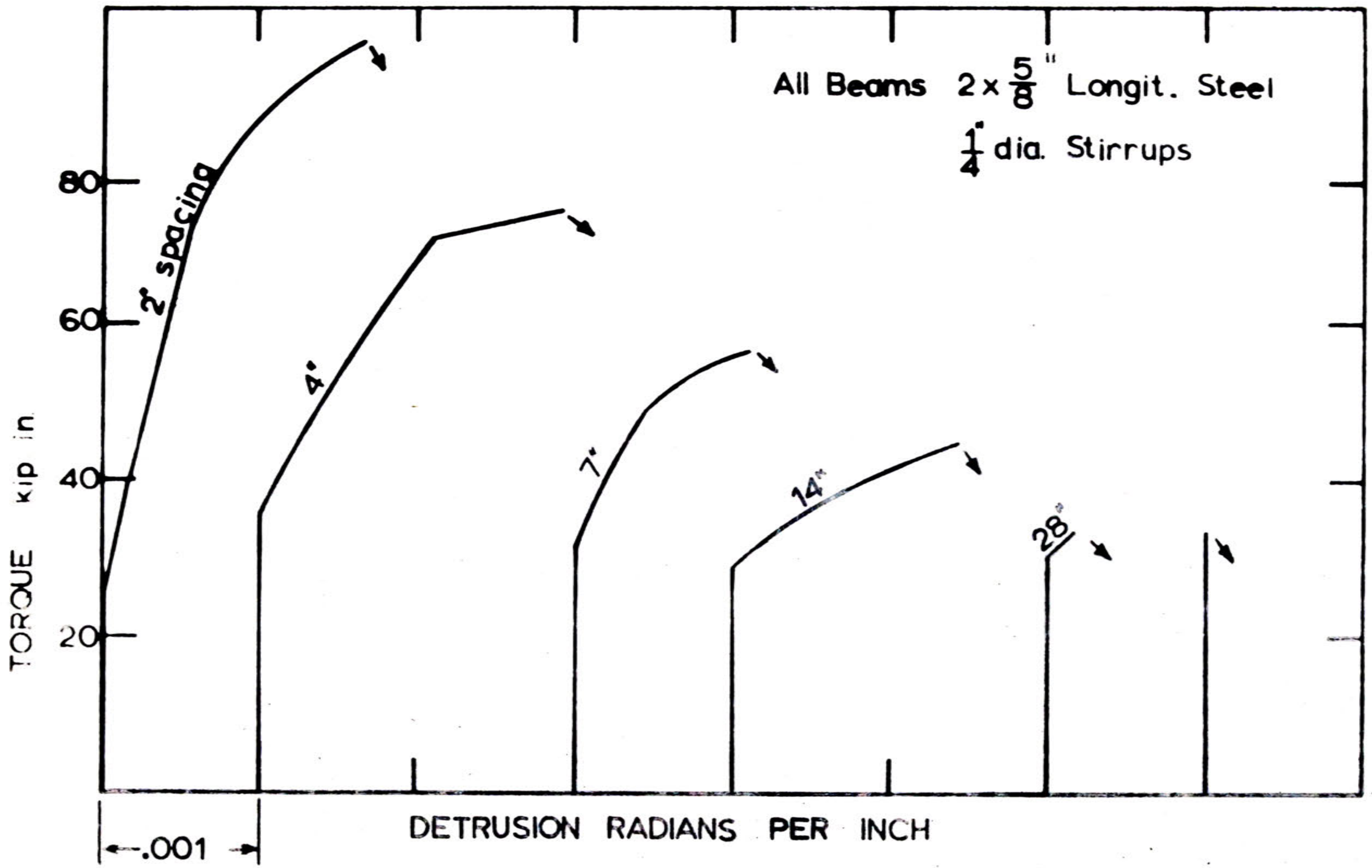


FIG.2.8 TORQUE TWIST CURVES FOR GROUP 3 OF ERNST'S TESTS.

2.3 (c) The Rotation and Deflection of Beams Loaded in Combined Bending and Torsion.

Nylander, who carried out extensive work on the strength of beams containing only longitudinal reinforcement, gave some attention to the rotational behaviour of concrete members. His tests on longitudinally reinforced beams showed that the presence of bending substantially reduces the torsional stiffness, although it should be noted that in these tests the full bending moment was applied prior to loading with torsion. Nylander tested two simple grid frames to justify the use of ultimate strength analysis methods for grids in which some members are subjected to torsional loading. The edge beams, which sustained torsional loading, were provided with web reinforcement. The results of these tests indicated that the members were sufficiently ductile to allow redistribution of forces in the frames.

In the work of Ramakrishnan and Vijarangan, a stated aim was to investigate "the relationship between torque and the angle of twist and how each factor influences this relationship". They reached the following conclusions for the post cracking behaviour of beams with only longitudinal reinforcement loaded in bending and torsion.

(i) The angle of twist increased as the concrete strength was reduced.

(ii) An increase in the percentage of tensile reinforcement reduced the angle of twist.

(iii) Compression reinforcement reduced the angle of twist.

(iv) The addition of bending moment reduced the angle of of twist.

The last conclusion contradicts the results obtained by Nylander. As the experimental evidence on which Ramakrishnan and Vijarangan based this conclusion was rather limited, it would be reasonable to place more confidence in the extensive work of Nylander.

Chinenkov, who carried out tests on web reinforced beams, gave the following qualitative conclusions on the subject of deformations.

1. Deflections in similar beams increase with increasing values of the ratio of torsion to flexure.
2. Transverse reinforcement affects the deflection when torsion is present.
3. Deflections decrease with increasing percentages of longitudinal reinforcement in the same manner as in pure flexure.
4. The effect of the concrete properties on deflection is greater than in the case of pure flexure.
5. For equal flexural moments the deflections under combined bending and torsion are higher than those calculated for pure flexure.
6. The rate of increase of the angle of twist increases with increasing loads.
7. For equal torsional moments the angle of twist is smaller for smaller flexural moments, for higher percentages of reinforcement and for higher strength of concrete.

CHAPTER 3

EXPERIMENTAL WORK

This chapter sets out details of the experimental work conducted in this investigation. For the purpose of discussion it is convenient to divide the tests into four groups, viz., plain concrete beams, beams containing only longitudinal steel, beams reinforced with both longitudinal and transverse steel tested primarily for ultimate strength, and web reinforced beams tested primarily for deformations.

Twenty-two plain concrete specimens were tested in torsion and combined flexure and torsion. As well as beams of rectangular section some cylindrical specimens were also tested. The main purpose of these tests was to determine whether the cracking load of a concrete section in bending and torsion could be satisfactorily predicted by employing a maximum stress criterion for concrete strength together with either "elastic" or the "plastic" distribution of stress across the section.

Thirty beams were cast and tested to investigate the behaviour and strength of beams containing only longitudinal steel subject to combined bending and torsion and combined shear and torsion.

Three series, totalling twenty-five beams, were cast with both transverse and longitudinal steel. These beams were then tested under combined loading mainly to check the accuracy of the proposed ultimate strength theory (Chapter 5). The first two series were designed to examine the effect of the ratio of top to bottom longitudinal steel on the torsion-flexure interaction curve. The third series was primarily designed to investigate the effect on the failure behaviour of varying the ratio of transverse to longitudinal steel. Tests of this series also compared tied and welded reinforcement, and open and closed stirrups.

Twelve beams containing web reinforcement were cast to investigate the deflections and rotations of beams designed in accordance with the proposed theory. (Chapter 7). These beams were more fully instrumented than those of the previous series.

3.1 TESTING METHODS

For the manufacture of the test specimens steel moulds and internal vibration were employed. Where desirable, a series of test beams was made from commercial transit mixed concrete. Such test series had the advantage that all beams had virtually the same concrete strength.

Curing was provided continuously up to testing. Most beams were cured under water, although some were cured by damp sand and hessian covered with plastic sheets. Comparative cylinder tests showed that the latter method was equally efficient.

Most of the beams were tested in a three-dimensional reaction frame. Figure 3.1 indicates diagrammatically how the loads were applied. Each beam was tested over an eight-foot span, one end being clamped against torsion and the other end being free to twist. The bending load was applied at the third points by means of a hydraulic jack and a spreader beam. The torsion was applied by another jack at the end of an outrigger arm. The jacks were hydraulically interconnected so that during the test the ratio of torsion to bending remained constant. The ratio of torsion to bending could be varied by changing either the jack size or the length of the outrigger. Special bearings under the spreader beam and under the ends of the specimen, ensured that the test beam was simply supported in bending and was restrained against twisting only at the fixed end.

For tests in which only a single point load was required the test rig shown in Figure 3.2 was used. The beams were loaded simultaneously in bending, torsion and shear by a single eccentric load. Again special bearings were employed to ensure that the member was simply supported in bending and was restrained against twisting only at the fixed end. Only the shear span near the fixed end was subjected

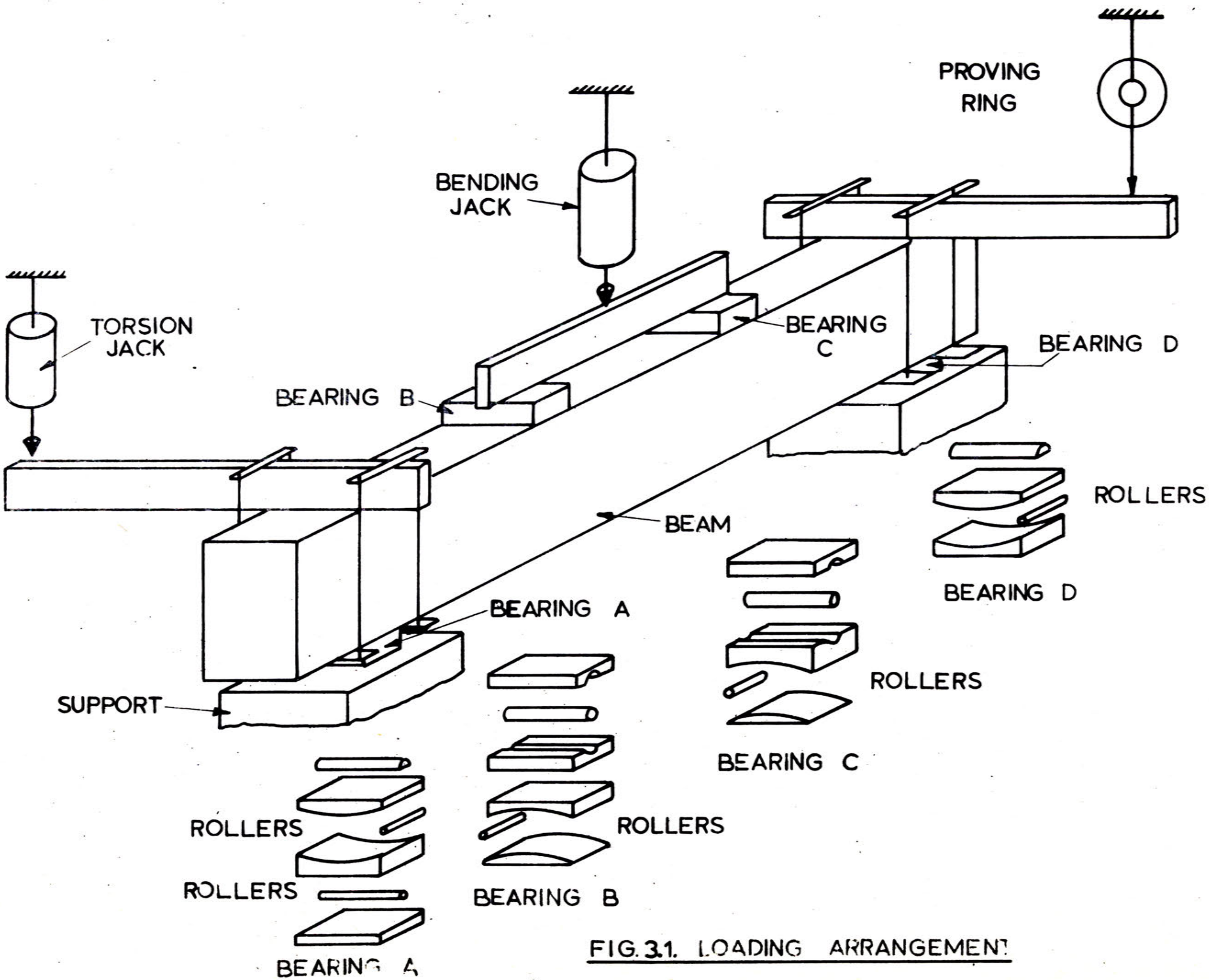


FIG. 3.1. LOADING ARRANGEMENT

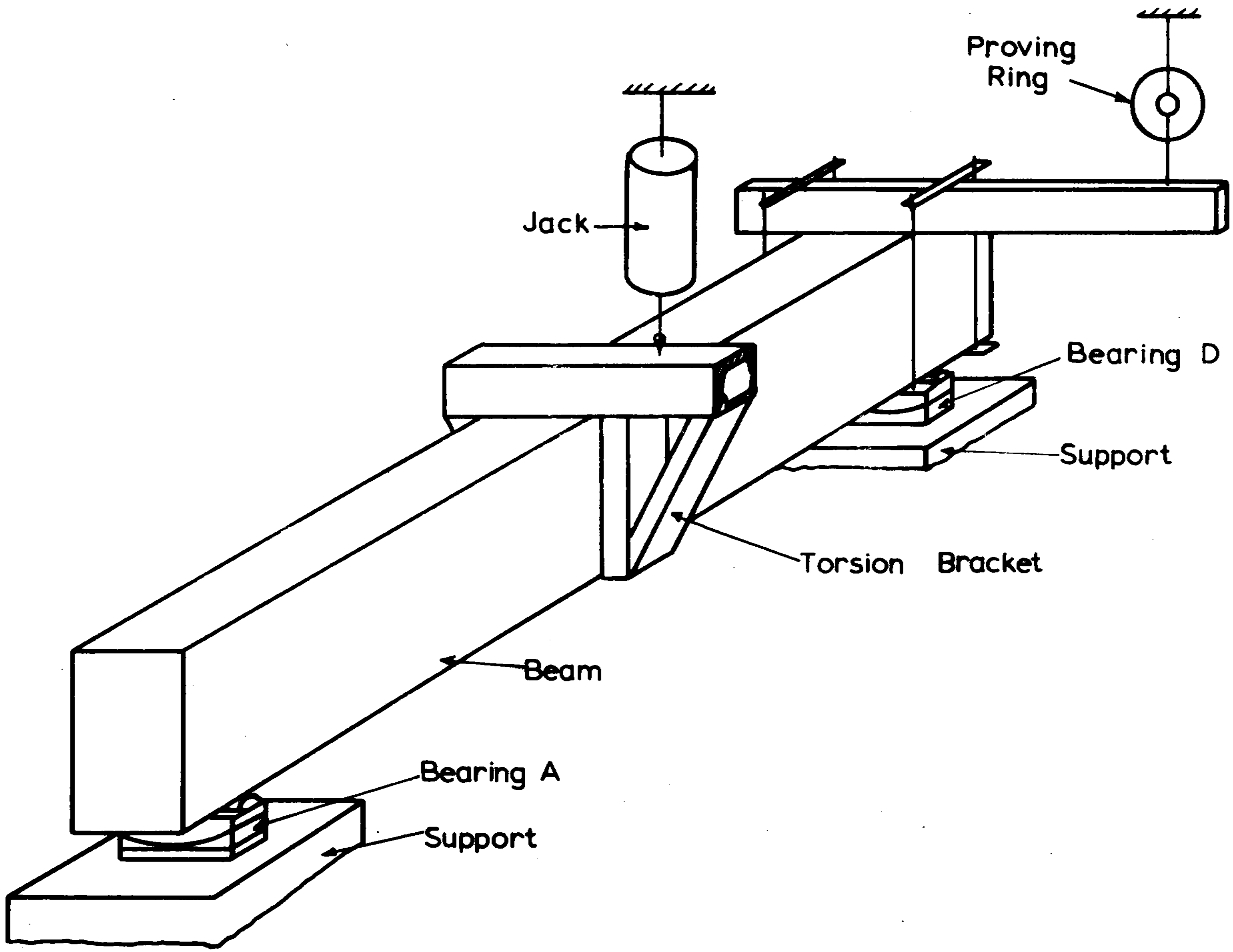


FIG. 3.2 ONE POINT LOADING RIG.

to torsion and the torque sustained was measured by a proving ring at the end of the outrigger arm on the fixed end.

The load was applied in about ten increments up to failure. Larger increments were applied in the initial stages of a test, and smaller increments as the loads approached the ultimate. The increments varied between 14% and 5% of the total load. After each increment, the load was held constant while crack development, deflections and rotations were recorded.

Rotation measurements were obtained by recording the deflections at the ends of rigid transverse arms.

To obtain continuous rotation measurements for the final test series a special rotation gauge was used. The design of this instrument was adapted from details of gauge reported by the Portland Cement Association. Essentially the gauge consists of a shaft which is fixed at one end to the beam, and a device for measuring the rotation of the other end of the shaft relative to the beam. A photograph of the gauge is given in Figure 3.3. An interesting feature is the use of bellows as couplings at each end of the shaft. These bellows act as universal joints and eliminate the effect of all displacements except rotation. The relative rotation between the shaft and the beam is translated into vertical movement which is detected by a modified inductance strain gauge. The impulses from the inductance are amplified and continuously recorded against the load.

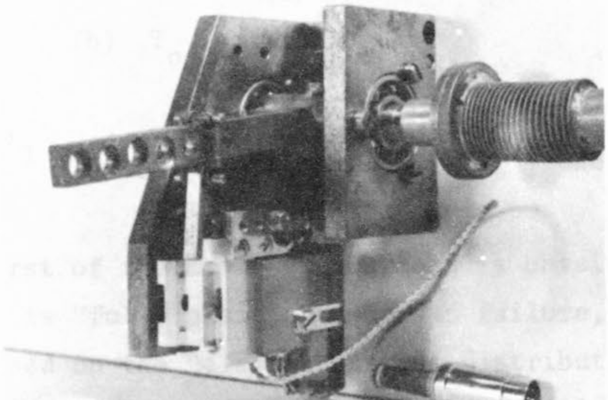


FIG. 3. 3 ROTATION GAUGE.

3.2 PLAIN CONCRETE BEAMS

3.2 (a) Background

If it is assumed that the concrete member subjected to pure torsion fails when the maximum principal tensile stress reaches the tensile strength of the concrete, then one of the following expressions for the torsional strength of a beam of rectangular section is obtained:-

$$\text{(a)} \quad T_o = \frac{b^2}{2} (h - b/3) \, f_t' \qquad \ldots\ldots \qquad (3.1)$$

or

$$\text{(b)} \quad T_o = K_1 b^2 h f_t' \qquad \ldots\ldots \qquad (3.2)$$

$$\text{where } K_1 = \frac{1}{3 + \dfrac{2.6}{0.45 + h/b}} \qquad \text{(approximately).}$$

The first of the above expressions is obtained if it is assumed that the section is "fully plastic" prior to failure, whereas the second expression is based on the "elastic" stress distribution. It is therefore apparent that different estimates of the tensile strength of the concrete in a given beam will be obtained from a torsion test by assuming "plastic" distribution on the one hand and "elastic" distribution on the other.

It has been found, however, that for a wide range of ratios of depth to breadth of a rectangular section the indicated tensile strength of the concrete, calculated on the basis of the failure torque with a "plastic" stress distribution is 0.6 ± 0.02 times the tensile strength calculated on the basis of an "elastic" stress distribution. Results of pure torsion tests on beams of rectangular cross section only, therefore, cannot provide sufficient evidence to support a theory of "elastic" stress distribution. Further, such tests cannot verify fully the maximum principal stress failure criterion.

A series of tests was therefore planned.

(a) to investigate the use of a maximum principal stress failure criterion,

(b) to test the suitability of the "elastic" and the "plastic" theories for calculating the cracking torque of any section.

(c) to collect additional information on the relationship between cylinder compressive strength and the torsional "tensile" strength.

3.2(b) Description of Specimens

Several relatively small test specimens were cast for testing in a conventional torsion machine (Tinius Olsen 300 kip-inches capacity). To attach the test specimens to the grips of the machine, steel end plates were glued to the specimens with a mixture of Epikote 818 and Polymid 65. Beams tested in this manner were HC, R1, SC, SCL and SCR. Details of these beams are given in Table 3.

To enable bending and torsion tests to be carried out, larger rectangular beams were cast. These beams were tested in the special rig described in the previous section. Details of these test beams are also given in Table 3.1.

The four specimens HC, R1, SC, and CRP1 were cast using a concrete mix containing 5/16" maximum size aggregate. The more usual 3/4" aggregate was used in the manufacture of beams P 1-6 and SCL. Commercial "ready mixed" concrete was employed in the casting of the series of beams containing web reinforcement and during the casting several plain concrete beams were also made. Thus beams REP2, REP4 and SCR were cast with the same concrete as beams of the RE series which will be discussed in a later section. Similarly, beams RUP2, RUP4 and ELL were cast with the RU series, W1 and W2 with two different mixes of the R series, W3 with the pour used for beams containing longitudinal steel. RP2 and W5 were cast as companion specimens with the deformation series and Q series respectively.

TABLE 3.1 - DETAILS OF PLAIN CONCRETE BEAMS

Beam	Size and Shape	Failure Loads (kip-in.)		Concrete Details		
		Torque	Moment	f'_c (p.s.i.)	f'_t (p.s.i.) Brazil	Type
HC	Hollow Cylinder O.D.=6"I.D.=4"	13.6	0.0	7,200	640	3/8" aggregate
R1	6"x4" Rectangle	13.4	0.0	"	"	"
SC	6" dia. Solid Cylinder	20.0	0.0	"	"	"
CRP1	9"x6" Rectangle	23.3	36.6	6,300	600	"
P1	9"x6" Rectangle	37.4	24.7	7,100	-	3/4" aggregate
P2	"	25.2	39.6	6,350	-	"
P3	"	47.6	5.5	6,550	970	"
P4	"	43.0	5.5	6,363	945	"
P5	"	0.0	51.3	7,450	880	"
P6	"	11.1	51.4	7,400	-	"
SCL	6" dia. Solid Cylinder	20.8	0.0	6,660	-	"
REP2	9"x6" Rectangle	29.6	20.3	4,600	575	Ready Mixed Concrete RE Pour
REP4	10" x $6\frac{1}{2}$" Rectangle	50.5	0.0	"	"	"
SCR	6" dia. Solid Cylinder	18.8	0.0	"	"	"

Table 3.1 Contd.

Beam	Size and Shape	Failure Loads (kip-in.)		Concrete Details		
		Torque	Moment	f'_c(p.s.i.)	f'_c(p.s.i.) Brazil	Type
RUP2	9"x6" Rectangle	19.0	35.7	3,680	516	Ready Mixed Conrete RU Pour
RUP4	10"x6½" Rectangle	66.3	0.0	"	"	"
ELL	9"x6" L-Shape 3" Thick	19.8	0.0	"	"	"
W1	10.4"x6.8" Rectangle	73.2	0.0	4,630	-	Ready Mixed Concrete R Pour
W2	10" x 6½" Rectangle	74.5	0.0	4,320	-	"
W3	10.2" x 6.9" Rect.	77.4	0.0	4,050	-	Longitudinal Pour
W5	10.2" x 5" Rectangle	31.9	0.0	3,560	665	Q Series Pour
RP2	10.3" x 5.2" Rect.	35.6	0.0	3,460	700	Deformation Pour

The beams were poured in steel moulds and cured under water for at least four weeks prior to testing.

3.2(c) Description of the Test Behaviour

All plain concrete beams failed immediately on the formation of a tensile crack approximately normal to the direction of the principal stress. Thus in pure torsion a spiral crack at 45^{o} to the longitudinal axis caused failure, whilst under bending and torsion the tensile crack formed at a steeper angle. The appearance of a typical failure surface is shown in Figure 3.4. Of course in all cases, the tension crack crossed only three sides of the beam, and a straight crack then joined the ends of this spiral across the fourth side.

3.2(d) Torque-Twist Curves

The torque-twist curves show very nearly linear elastic behaviour almost up to failure. However a certain amount of inelastic behaviour seems evident just before failure (see Figure 3.5). As the elastic deformations of a reinforced section are approximately the same as those of a plain section, the stiffness of plain sections is of particular interest.

The results obtained for the shear modulus are given in Table 3.2.

TABLE 3.2

EXPERIMENTAL SHEAR MODULUS OF ELASTICITY FOR PLAIN BEAMS

Beam	f'_c p.s.i.	G Experimental p.s.i. x 10^6
P1	6,400	2.10
P2	6,350	2.16
P3	6,550	2.08
P5	7,400	2.04
REP4	4,600	1.71
RUP4	3,680	1.50
W1	4,630	1.86

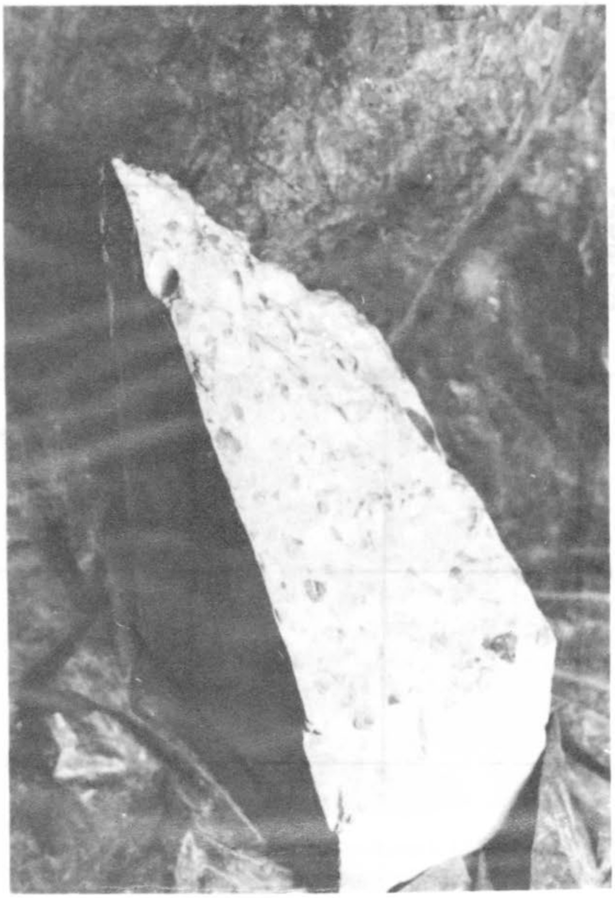


FIG. 3.4. FAILURE VIEW OF PURE TORSIONAL PLAIN CONCRETE SPECIMEN.

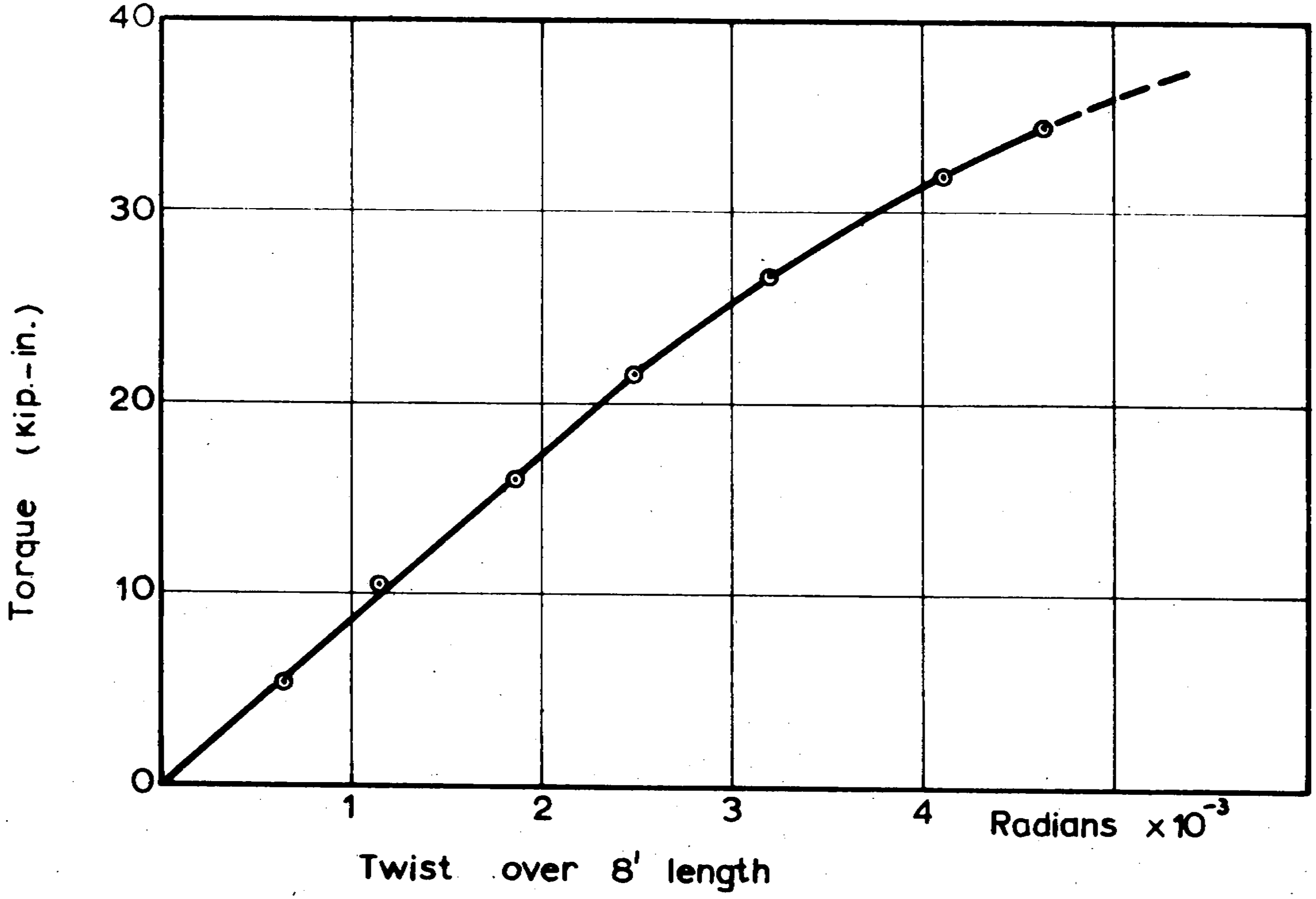


FIG 3.5 TORQUE – TWIST CURVE FOR PLAIN CONCRETE
(BEAM P1)

3.3 BEAMS CONTAINING ONLY LONGITUDINAL REINFORCEMENT

3.3(a) Background

In the review of the published works on torsion the following conclusions were arrived at:-

1. It is reasonable to ignore the reinforcement in calculating the pure torsional strength of beams containing only longitudinal steel.
2. The effect of shear can be accounted for approximately by the following formula -

$$T/T_o = 1 - V/V_o \qquad \ldots\ldots \qquad (3.3)$$

where

T_o = the pure torsional strength.

V_o = the calculated shear strength of the member in simple flexure. Unless a more detailed analysis is made this can be taken as

$2.0 \sqrt{f'_c}\, bd$.

3. No satisfactory method has been proposed for calculating the ultimate strength of beams of this type under combined bending and torsion.

A series of tests was carried out to investigate the behaviour and strength of the beams under combined bending and torsion and twenty one beams were tested to confirm the safety of the shear recommendations.

3.3(b) Description of Specimens

All test beams were rectangular in section and were ten feet long. Full details of the beams are given in Figure 3.6 and Table 3.3. Curing was provided for at least 28 days prior to testing. Where bending and torsion failures were required (beams L1, L2, L3, L4, L7, L8, LB1 and LB3) the shear spans were reinforced with 3/8" ties at 3" centres.

TABLE 3.3 DETAILS OF LONGITUDINALLY REINFORCED BEAMS
PART 1 BEAMS SUBJECTED TO TWO POINT LOADING

Beam	Geometry of Beam (inches)			Steel				Concrete	Failure Loads		
	b	h	d	A_{L1} sq. in.	f_{L1} kips/ sq. in.	A_{L3} sq. in.	f_{L3} kips/ sq. in.	f'_c p.s.i.	Torque kip. in.	Moment kip. in.	Shear kips.
L1	6	9	7.9	0.88	45.0	-	-	6360	57.0	32.3	-
L2	6	9	7.9	0.88	45.0	-	-	6580	60.1	205.7	-
L3	6	9	7.9	0.88	45.0	-	-	7225	61.1	5.5	-
L4	6	9	7.9	0.88	45.0	-	-	7260	0.0	276.0	-
L7	6	9	7.9	0.88	45.0	-	-	6190	47.8	80.5	-
L8	6	9	7.9	0.88	45.0	-	-	6470	49.6	263.5	-
L5	6	9	7.9	0.88	45.0	-	-	6400	43.4	144.4	4.05
L6	6	9	7.9	0.88	45.0	-	-	6780	52.5	29.7	0.75
SL1	6	9	7.9	0.88	45.0	-	-	3300	46.2	77.3	2.20
LS1	6	9	7.6	0.88	45.0	-	-	4050	-	252.0	8.98
LS 2	6	9	7.6	0.88	45.0	-	-	4050	52.5	5.2	
LS 3	6	9	7.6	0.88	45.0	-	-	4050	39.6	53.4	1.88
LS 4	6	9	7.6	0.88	45.0	-	-	4050	25.7	117.8	4.18
LS 6	6	9	7.6	0.88	45.0	-	-	4050	37.6	101.7	3.56
S1	6.5	10.0	9.2	0.91	41.0	-	-	4050	41.8	181.0	6.40
S2	6.5	10.0	9.2	0.91	41.0	-	-	4050	-	310.0	11.05
S3	6.5	10.0	9.2	0.91	41.0	-	-	4050	64.2	6.3	
S4	6.8	10.2	9.3	0.91	41.0	-	-	4050	47.0	126.9	4.50
S5	6.5	10.0	9.2	0.91	41.0	-	-	4050	47.7	69.3	2.43
LB1	6.5	10.0	8.9	0.91	41.0	0.61	41.0	4050	54.1	141.6	-
LB2	6.5	10.0	9.1	0.91	41.0	0.61	41.0	4050	55.4	6.3	-
LB3	6.5	10.0	8.9	0.91	41.0	-	-	4050	60.7	161.5	-

TABLE 3.3

PART 2 BEAMS SUBJECTED TO ONE POINT LOADING

Beam	f'_c p.s.i.	Nominal a/d	Nominal e/b	Failure Loads T kip. in.	M kip. in	V kips.
Q1	3,560	5.5	0.3	13.9	263	5.60
Q2	"	5.5	0.6	24.1	193	4.20
Q3	"	5.5	0.0	0.0	306	6.45
Q4	"	5.5	1.8	27.8	72	1.58
Q4a	"	-	∞	33.8	0	0.0
Q5	"	4.1	0.3	16.5	205	5.75
Q6	"	3.0	0.3	19.4	162	6.05
Q6a	"	3.0	0.6	24.4	114	4.27
Q7	"	4.1	1.8	28.8	60	1.71
Q7a	"	4.1	0.6	27.3	154	4.32
Q8	"	3.0	0.0	0.0	226	8.40
Q8a	"	3.0	1.8	33.0	47	1.80
Q9	"	5.5	0.3	18.3	237	5.[illegible]
Q10	"	2.0	1.8	36.4	39	2.20
Q11	"	2.0	0.0	0.0	225	12.55
Q11a	"	2.0	0.3	26.5	72	4.05
Q11b	"	2.0	0.6	27.2	180	10.05
Q12	"	4.1	0.0	0.0	271	7.55

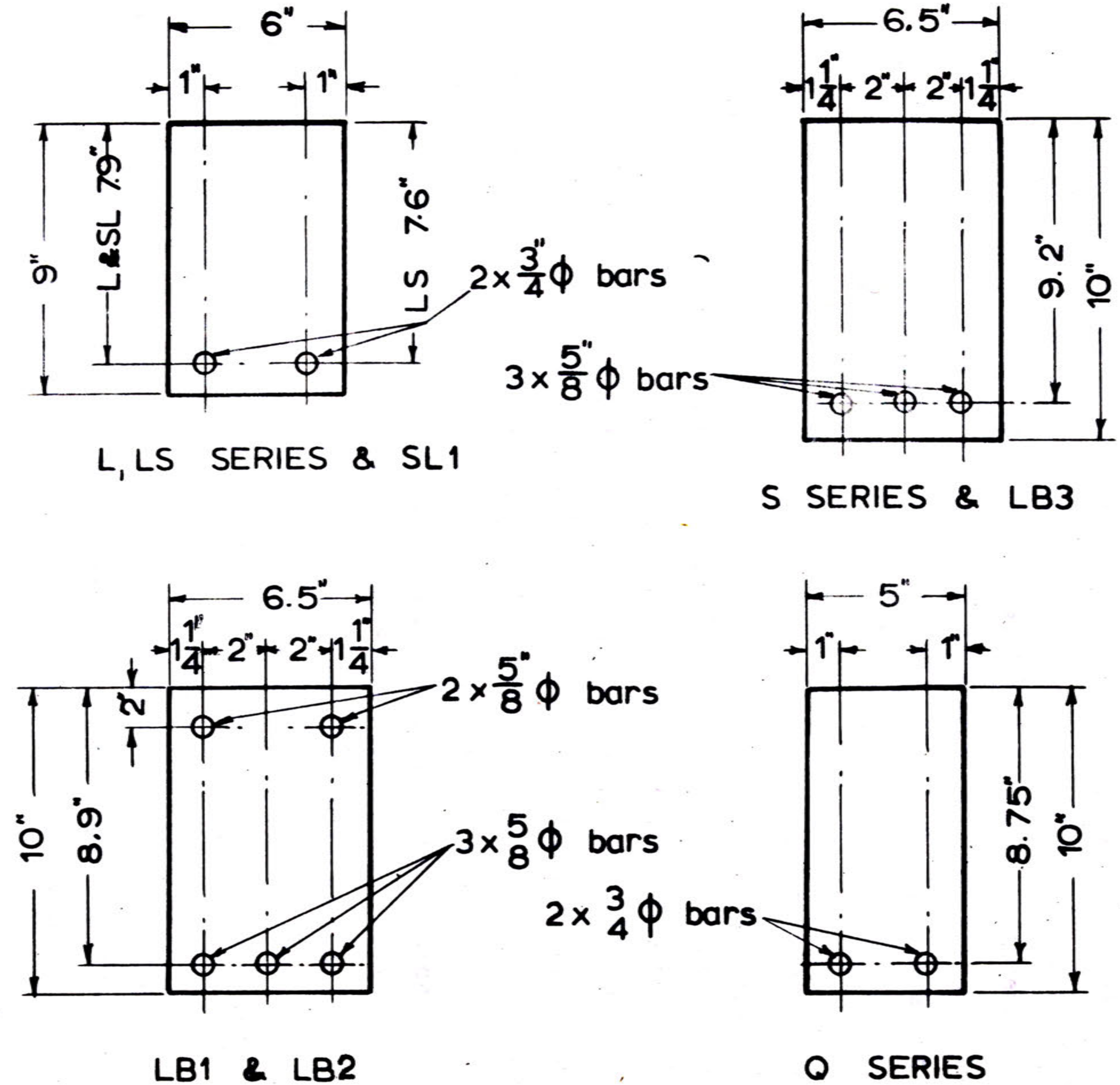


FIG. 36 DETAILS OF BEAMS CONTAINING LONGITUDINAL STEEL.

The concrete mix proportions for beams L1 - L8 were the same as used in the beams of the plain concrete series P1 - P6. All other beams were poured from ready mixed concrete. The one batch was used for beams of the LS, LB and S groups.

3.3(c) Description of Tests

The behaviour of all beams up to cracking was essentially elastic. Consideration of the torsional stiffness of the beams at this stage suggests that the stiffness is unaffected by the reinforcement, and may be taken as that of a plain concrete section. For instance the shear modulus of elasticity for beams L1 and L2, if the reinforcement is ignored, is 1.93×10^6 p.s.i. and 2.46×10^6 p.s.i.

For beams loaded in combined bending and torsion, two distinct types of behaviour were observed. Under predominantly torsional loading ($\psi > 0.5$ approximately) a brittle failure occurred. This type of failure took place in beams L1, L3, L6, L7, LB1, LB2 and LB3. Cracks were visible only just before failure. The cracks were inclined to the longitudinal axis of the beam, being approximately normal to the direction of the principal tensile stress. Initially cracks occurred in the lower portion of the cross section but were prevented from opening by the bottom longitudinal steel. With increasing load the torsional stresses probably increased more rapidly than the bending compressive stresses following the reduction of the effective cross-section by cracking. A stage was eventually reached when an inclined tensile crack extended to the upper surface of the beam resulting in a sudden drop in the torsional resistance. The torque on the section was then resisted by dowel forces and further rotation took place at greatly reduced loads. Figure 3.7 shows the torque-twist behaviour of a beam (L1) failing in this manner, and Figure 3.8 the appearance of such a beam at failure.

If the test beam was loaded at lower ratios of torsion to

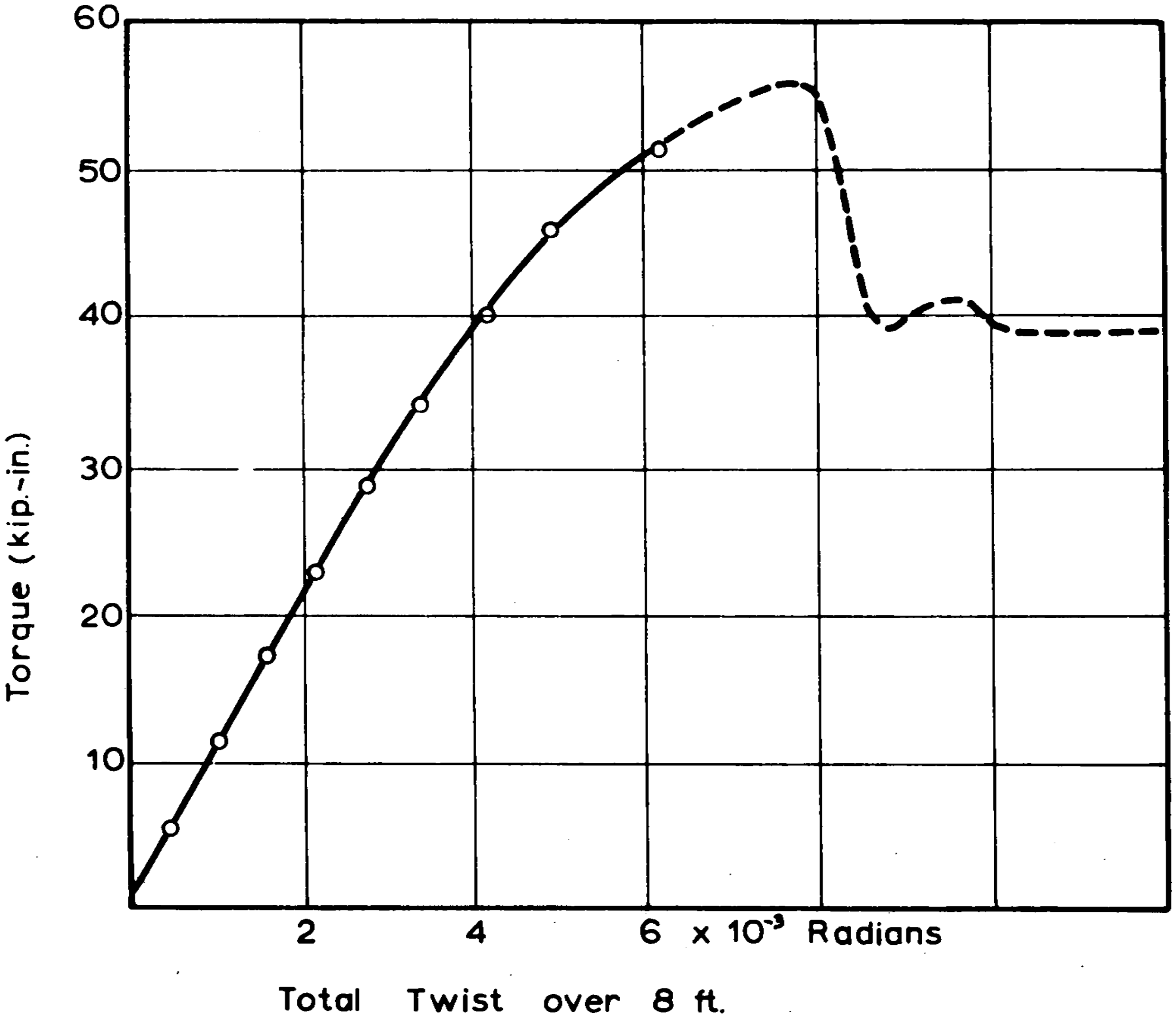


FIG. 3.7 TORQUE—TWIST CURVE FOR BEAM L 1

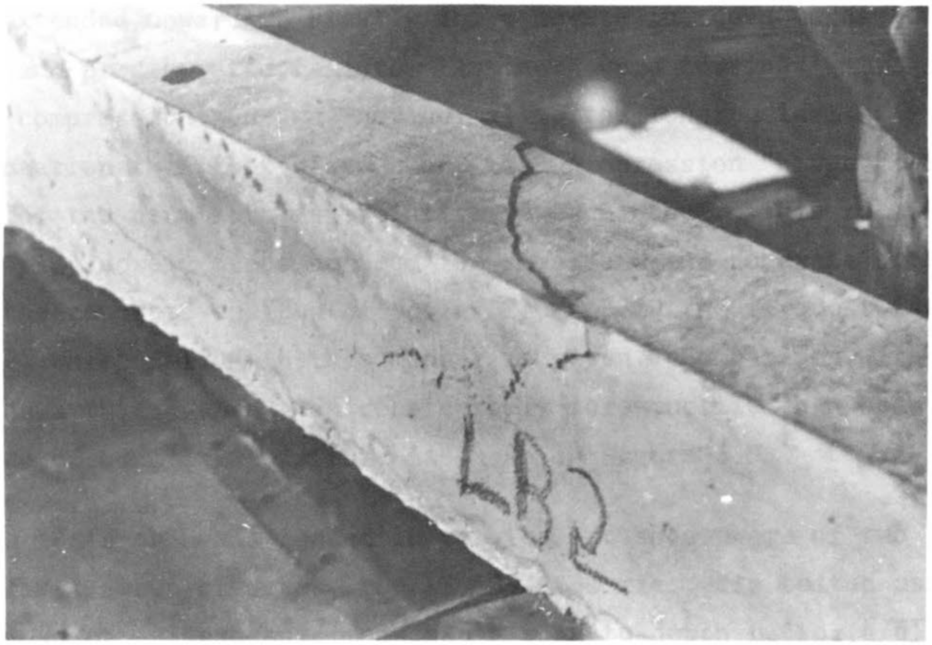


FIG. 3.8. FAILURE OF BEAM CONTAINING ONLY LONGITUDINAL STEEL LOADED IN PREDOMINANTLY TORSION.

bending (ψ < 0.5) a more gradual failure ensued which was similar to a flexural failure. Beams L2 and L8 failed in this manner. As before, a cracked zone developed in the lower portion of the cross section. In this case, however, the higher bending moment prevented the cracks from spreading to the top surface of the beam. With increased loading the tension cracks widened until the steel yielded, whereupon the cracks opened further and extended upwards. Finally the concrete in the compression zone crushed and precipitated failure. As there were torsional stresses present, the compression zone was not normal to the longitudinal axis of the beam. Rotation then took place about this compression "hinge", with the two portions of the beam rotating relative to each other. This rotation brought into play dowel forces between the steel and the concrete until large segments of the concrete spalled off. In Figure 3.10 the torque-twist curve of a beam (L2) which failed in this manner is given. Comparison with Figure 3.7 shows that beam L2 was considerably more ductile than beam L1. The appearance of beam L2 at failure is shown in Figure 3.9.

The tests on beams loaded in torsion and shear were of two types. The first group of beams, the L5, S, SL series were tested using third point loading. These specimens had a span to depth ratio (a/d) of approximately 3 and were tested at high ratios of torsion to shear. To investigate the effect of varying a/d and to consider lower ratios of torsion to shear the Q series of beams were tested. One point loading was used for this series. As the test span for some specimens was as low as 18" it was possible to obtain multiple tests on some of the 10 ft. beams.

Under predominantly torsional loading failure occurred upon the formation of a diagonal tension crack. This tension crack spread to the top and the bottom of the beam forming a spiral. A well defined compression zone then developed on the fourth side. The appearance of a beam failing in this manner is shown in Figure 3.11.

As the proportion of shear in the loading increases a more gradual failure ensues, and the ability of the beam to sustain diagonal

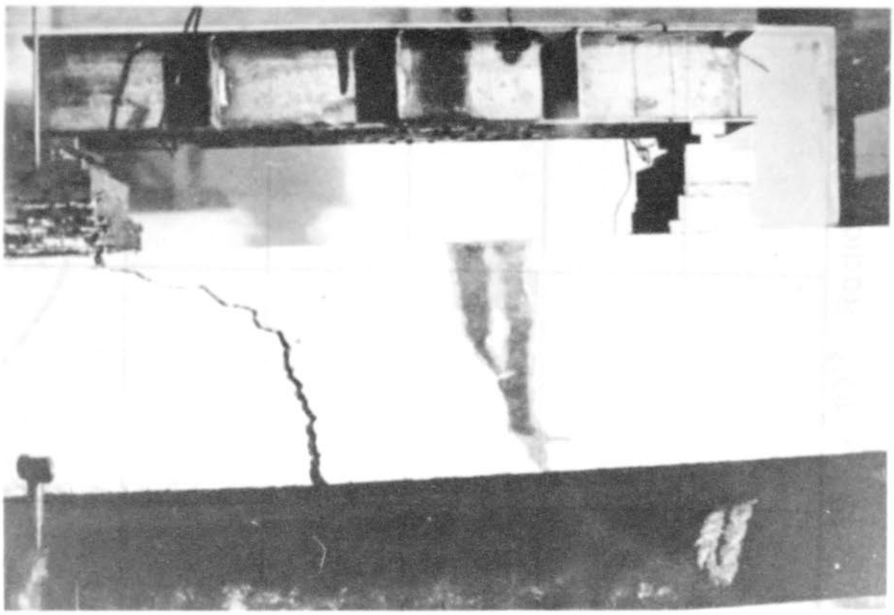


FIG. 3.9. FAILURE OF BEAM CONTAINING ONLY LONGITUDINAL STEEL LOADED IN PREDOMINANTLY BENDING.

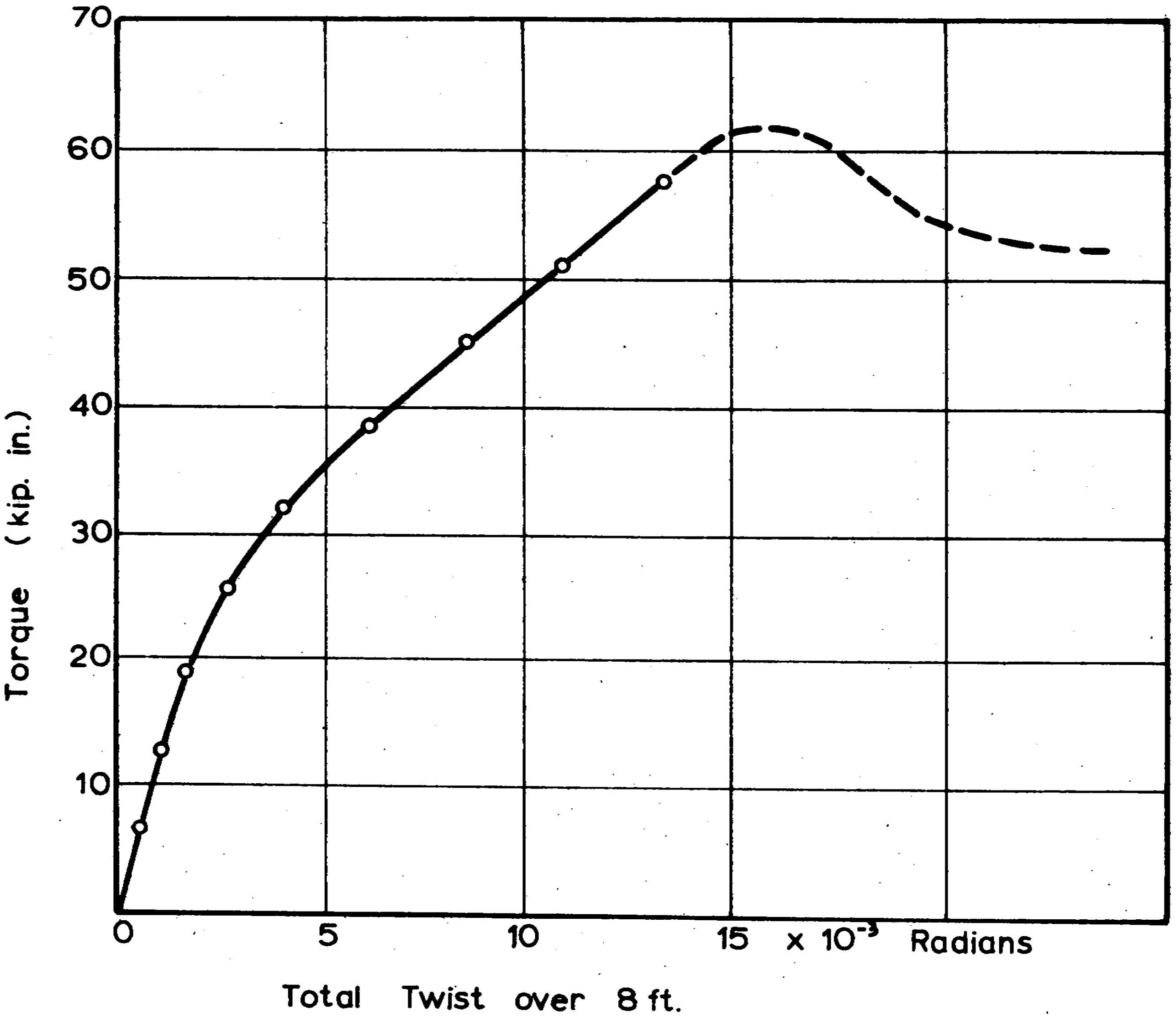
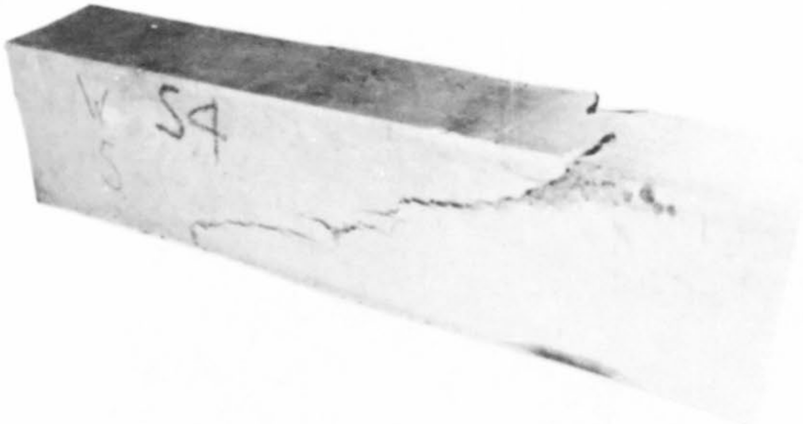
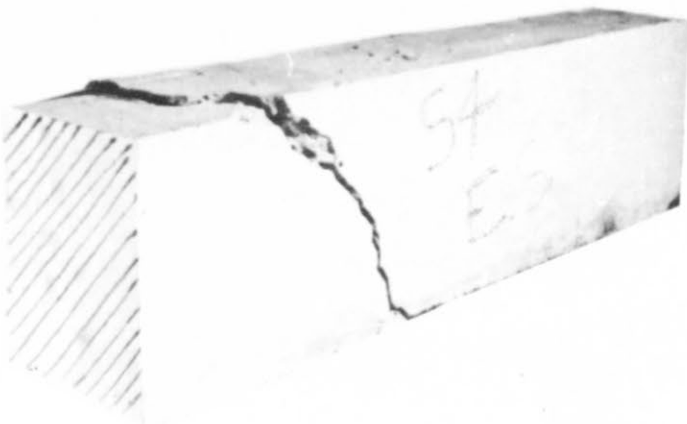


FIG. 3.10 TORQUE - TWIST CURVE FOR BEAM L 2.



SIDE ON WHICH TORSION OPPOSES SHEAR



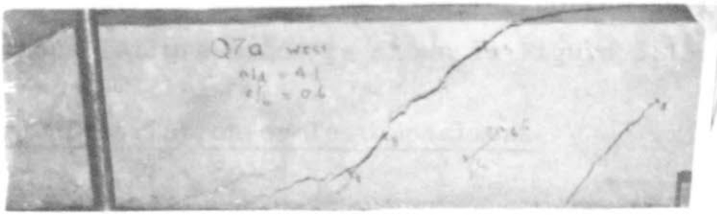
SIDE ON WHICH TORSION AND SHEAR ARE ADDITIVE

FIG. 3.11. THE APPEARANCE OF BEAM S4 AT FAILURE

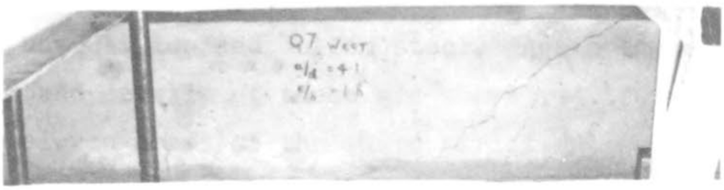
cracking increases. In Figure 3.12 the sides, on which torsional and shear stresses are additive, of four beams are shown. It can be seen that the resistance of the beams after cracking and the number of cracks at failure is increased with increasing ratios of torsion to shear.

Q 12 $\frac{a}{d} = 4.1$ $\frac{e}{b} = 0.0$

Q 5 $\frac{a}{d} = 4.1$ $\frac{e}{b} = 0.3$



Q 7a $\frac{a}{d} = 4.1$ $\frac{e}{b} = 0.6$



Q 7 $\frac{a}{d} = 4.1$ $\frac{e}{b} = 1.8$

FIG. 3.12. THE EFFECT OF THE RATIO $\frac{e}{b}$ ON THE APPEARANCE AT FAILURE

3.4 BEAMS CONTAINING BOTH LONGITUDINAL AND TRANSVERSE STEEL

An ultimate strength method for the analysis and design of this type of beam is presented in a later chapter. In this method several possible modes of failure are considered. When failure is caused by yielding of the reinforcement, it is assumed that the beam will fail in one of three idealized modes.

In each of the modes a cracked tensile zone intersects three exterior faces of the beam in a rectangular helix while a compression zone near the fourth face joins the two ends of this helix. The beam is assumed to fail when the steel intersected by the tensile crack yields, allowing the beam to rotate about an axis in the compression zone. Failure with this axis near the top surface of the beams is referred to as a mode 1 failure, near the side face as a mode 2 failure, whilst a mode 3 failure indicates that the axis forms near the bottom surface. The appearance of these failure modes is shown in Figure 3.13.

3.4(a) Description of Test Specimens

The first series (RE) comprised five beams having equal top and bottom longitudinal steel. The specimens were similar and details are given in Figure 3.14 and Table 3.4. The second series (RU) comprised nine beams with unequal top and bottom steel. Again these specimens were all similar and details of these are shown in Figure 3.14 and Table 3.4. The eleven beams of the third series (R) were not all similar and their details are given in Figure 3.14 and Table 3.5.

The reinforcement cages were prefabricated from plain round bars. For beams of the first two series the transverse steel was spot-welded to the longitudinal steel while for beams of the third series it was tied. The yield strengths of the bars used in the first two series were as follows:- 49 kips/sq.in. for No. 3 bars; 44.5 kips/sq.in. for No. 4 bars; 46.8 kips/sq.in. for No. 6 bars. The yield strengths of the bars used in the third series are given in Table 3.4.

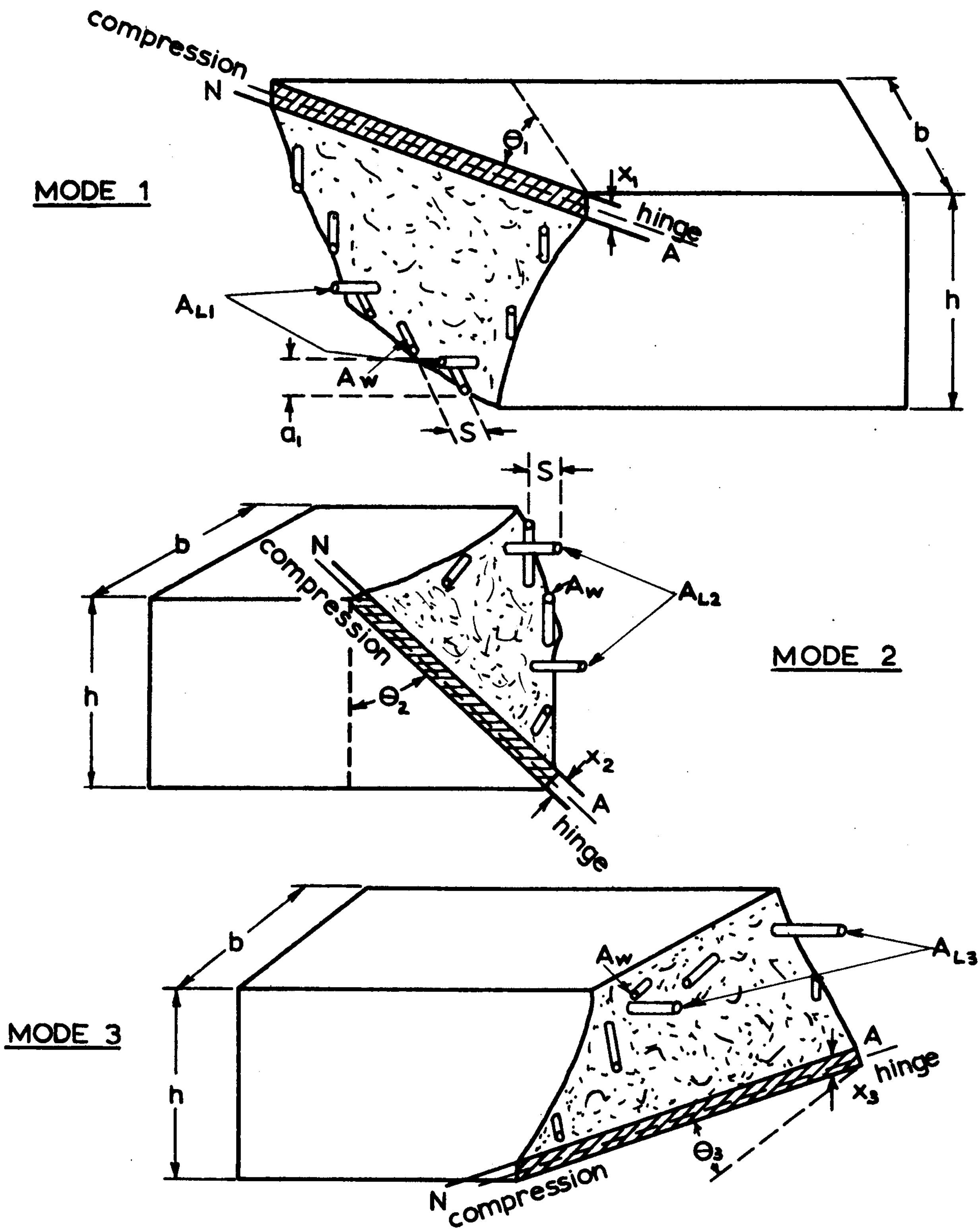
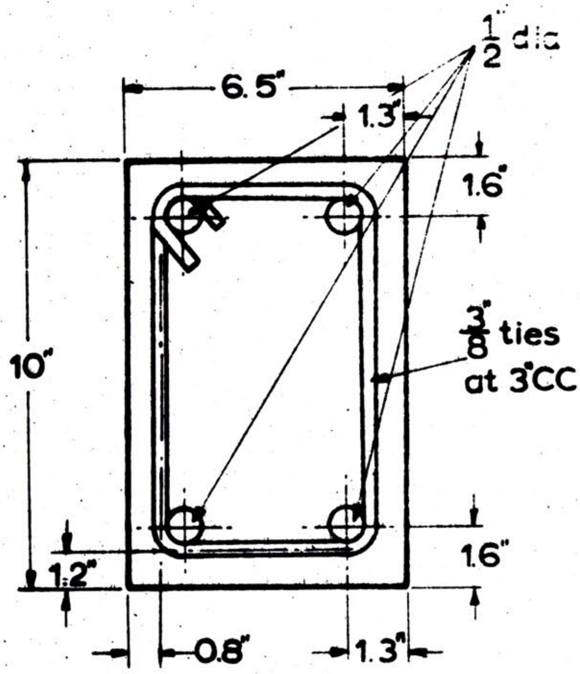
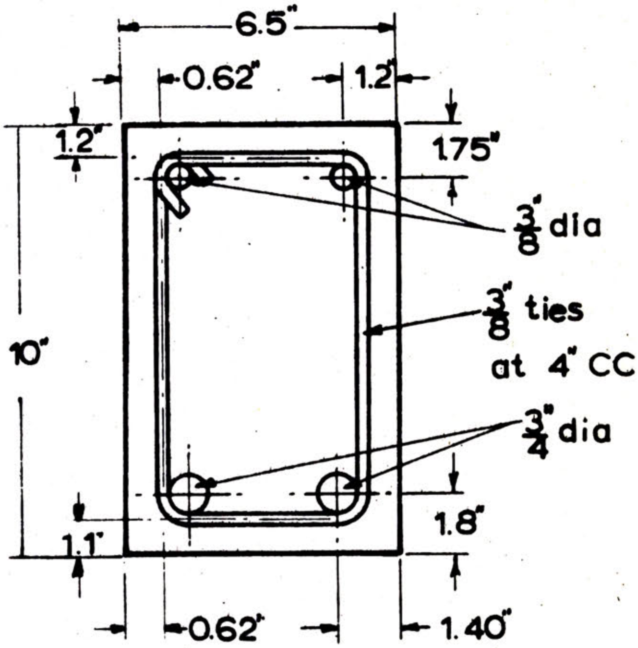


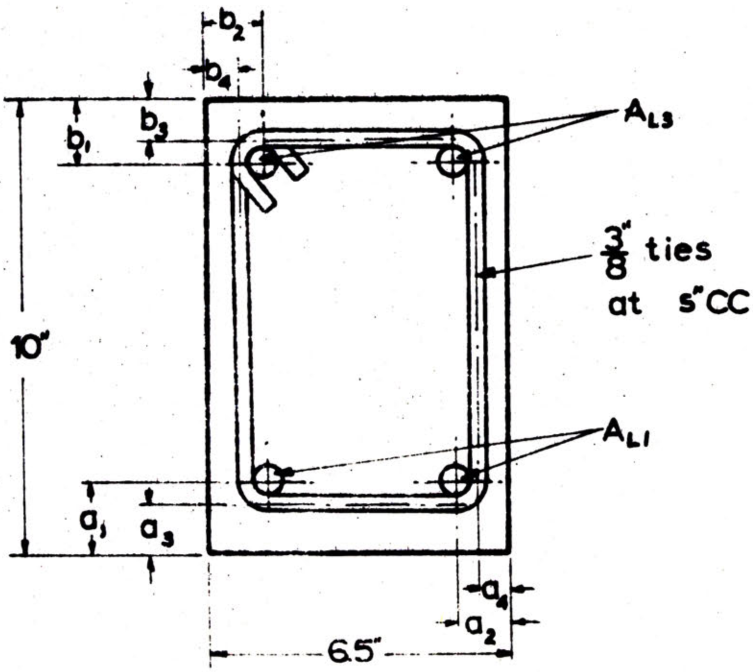
FIG. 3.13 IDEALIZED FAILURE MODES



RE SERIES



RU SERIES



R SERIES

TABLE 3.4

DETAILS OF BEAMS OF RE AND RU SERIES

(See also Figure 3. 14)

Beam	ψ	Experimental failure loads			Observed Failure Mode	f'_c p.s.i.
		T (kip-ins.)	M (kip-ins.)	V (kips)		
RE1	12.90	81.4	6.3	0.3	2	4,600
RE2	2.61	83.4	32.0	0.9	1	"
RE3	1.80	81.5	45.3	1.3	1	"
RE4	0.88	74.6	84.4	2.4	1	"
RE5	0.61	66.0	108.2	3.1	1	"
RE4*	0.28	38.0	134.0	3.8	1	"
RE2*	0.00	0.0	160.0	4.6	-	"
RU1	11.60	73.3	6.3	0.3	3	3,680
RU3A*	12.08	76.0	6.3	0.3	3	4,630
RU2**	1.66	84.9	51.1	-	3	3,680
RU3**	1.25	105.0	84.0	-	3	3,610
RU3A**	0.63	89.4	141.3	-	1	4,630
RU4	0.59	85.5	145.0	4.1	Indefinite	3,680
RU5	0.30	75.4	249.7	7.1	1	3,680
RU5A	0.25	68.3	266.8	7.6	1	4,400
RU6	0.21	59.1	281.2	8.0	1	3,610
RU7	0.00	0.0	304.0	8.7	-	3,680

* Retest
** Shear Spans Clamped

TABLE 3.5 DETAILS OF R SERIES
PART 1 BEAM PROPERTIES

					Longitudinal Steel				Transverse Steel		
Beam	a_1	a_2	a_3	a_4	A_{L1} sq. in.	f_{L1} kips/sq. in.	A_{L3} sq. in.	f_{L3} kips/sq. in.	s ins.	f_x kips/sq. in.	Type
36T4	1.6	1.3	0.8	0.6	0.88	37.7	0.22	43.4	4	43.0	Closed
36T4c	1.6	1.4	0.8	0.7	0.88	37.7	0.22	46.7	4	43.0	Closed
36T5.5	1.6	1.4	0.9	0.7	0.88	37.7	0.22	43.4	$5\frac{1}{2}$	43.0	Closed
77T5	1.7	1.4	0.9	0.6	1.20	38.8	1.20	38.8	5	43.0	Closed
7705	1.5	1.3	0.7	0.5	1.20	38.8	1.20	38.8	5	43.0	Open
77T4	1.7	1.4	0.9	0.6	1.20	38.8	1.20	38.8	4	43.0	Closed
7704	1.7	1.4	0.9	0.6	1.20	38.8	1.20	38.8	4	43.0	Open
7703	1.7	1.6	0.9	0.8	1.20	38.8	1.20	38.8	3	43.0	Open
24T3	1.3	1.4	0.8	0.9	0.39	47.1	0.10	68.7	3	43.0	Closed
38T5	1.7	1.6	0.9	0.8	1.57	38.7	0.22	45.2	5	43.0	Closed
3304	1.4	1.4	0.9	0.9	0.22	43.0	0.22	43.0	4	43.0	Open

TABLE 3.5

PART 2. FAILURE LOADS

Beam	ψ	Experimental Failure Loads			Observed Failure Mode	f'_c p.s.i.
		T (kip-ins.)	M (kip-ins.)	V (kips)		
36T4	0.26	62.6	240.4	7.5	1	4,400
36T4c**	1.54	94.1	61.1	-	3	4,340
36T5.5**	0.50	85.9	173.4	-	1	4,630
77T5	0.35	91.6	262.4	8.2	Indefinite	4,630
7705	0.35	96.4	278.4	8.7	Indefinite	4,630
77T4	0.48	107.6	223.4	7.0	1	4,630
7704**	0.49	99.2	201.4	-	Indefinite	4,630
7703	0.34	74.6	218.9	6.8	Indefinite	3,830
24T3**	1.52	70.8	46.6	-	3	4,340
38T5	0.37	80.1	216.4	6.7	2	3,830
3304	0.32	30.3	94.4	3.0	1	3,830

** Shear Spans Clamped

Commercial transit mixed concrete with a nominal strength of 3,000 p.s.i. and a slump of 3 inches was used. The aggregate consisted of 3/4 inch round river gravel, a river sand and a fine beach sand. The cylinder crushing strength of the concrete at the time of testing is recorded in Tables 3.4 and 3.5.

3.4(b) Test Behaviour

In each of the test series RE and RU the first specimen was tested with a high ratio of torsion to bending and this ratio was reduced for each successive member of the series. The ratio (ψ) of torsion to flexure is shown in Table 3.4.

In all beams with high ratios of torsion to bending (i.e. specimens RE1, 2, 3 and RU1, 2, 3, 3A), cracks were first visible on the side surfaces. The cracks formed at about 45° to the axis and they gradually extended, at almost constant inclination, to the top and bottom of the member. Upon increasing the load the tensile cracks spread across the bottom surface of the beam, and at still higher loads appeared on the top surface. Failure of the beam ensued when the longitudinal steel yielded, permitting opening of the tensile cracks on three sides of the beam and rotation of the member about an axis near the fourth side.

Beams RE2 and RE3 (with equal top and bottom steel), failed by yielding of the bottom longitudinal steel and opening of the tensile cracks on the sides and bottom of the beams. That is to say, these beams failed in mode 1. The beams with more steel in the bottom (RU1, 2, 3, 3A), on the other hand, failed by yielding of the top steel and opening of tensile cracks on the sides and top of the beams. This has been defined as a mode 3 failure.

This behaviour would be anticipated for beams sustaining nearly pure torsion. For a symmetrically reinforced beam (RE), even a small amount of bending will cause the final failure to take the form of mode 1. If the area of the top steel is much less than that of the bottom steel, the top steel is more likely to yield and a mode 3 failure will result, with a compression hinge at the bottom, unless counteracted by the presence of quite high bending moments.

The specimens with lower ratios of torsion to bending (i.e. RE4, 5, and RU5, 5A, 6, 7) were clearly influenced primarily by flexure. Cracks were observed first in the bottom faces of the beams. At higher loads these cracks extended to the side surfaces where their angle changed from almost vertical at the bottom to approximately 60° to the vertical near the top. For beams of both series, the predominance of bending forced the compression hinge to occur on the top (mode 1).

Beam RE1, whose initial cracking was described previously, failed with the compression hinge near a side face (mode 2). Usually this type of failure is associated with the presence of shear force. In this beam, the side cover to the main reinforcement was greater than for the other beams of the series (1.7 in. instead of 1.3 in.), and the specimen was subjected to almost pure torsion.

Beam RU4 failed in the shear span and its mode of failure was not clearly defined. Subsequent calculation showed that the three failure modes were nearly evenly balanced at this point. Because of this, specimens RU2, 3 and 3A (tested later than RU4) were provided with additional external clamps in the shear spans to ensure their failure in the central region.

Beams RE4, RE2 and RU3A were unloaded immediately upon their reaching failure load. The ratio of torsion to flexure in each case was then changed and, in the case of RU3A, the external clamps were shifted from the shear spans to the central region. The specimens were then reloaded. All three beams developed 'new' cracks during the reloading and developed 'new' fracture surfaces at failure. The results of these re-tests along with the earlier beams are given in Table 3.4.

In the third test series, three beams (36T4, 36T5.5 and 3304) which were tested at low ratios of twisting to bending moment, exhibited failure modes definitely of the mode 1 form described earlier. For each of these beams, cracks were first observed on the bottom surface at about 70% of the failure load. These cracks then appeared on the sides, and as the load was increased, they gradually spread up the sides changing their inclinations from about 10° to the vertical near the bottom to about 45° in the middle and to about 60° to the vertical near the top. At still higher loads, one or more of the tension cracks began to open. As the beam rotated, shattered compression zone became plainly visible on the top face. The load then began to drop off gradually, showing that failure had occurred. With additional rotation, the main cracks widened and horizontal cracks probably due to dowel forces appeared about two inches below the top face. The appearance of the failure surface at this stage is shown in Figure 3.15.

The fact that the transverse steel was tied to the longitudinal steel in this series and not welded as in the earlier series, made no appreciable difference to the behaviour of the beams. Thus beam 36T4, which was similar to beam RU5A and was loaded under similar conditions, exhibited a failure crack pattern closely resembling that of beam RU5A (compare Figures 3.15 and 3.16).

Those beams which contained large size longitudinal bars (beams 77T5, 77T4, 7704 and 7703) failed in a relatively brittle manner. That is to say, once the failure torque was reached, the load dropped off suddenly with very little 'plastic' rotation occurring. Furthermore, the modes of failure of these beams could not easily be classified.

Beams 77T5 and 7705 were identical except that the former was reinforced with closed ties while the latter had open stirrups. In both beams the spacing of the stirrups was 5 in. in the centre zone and 2½ in. in the shear spans. In both beams the first cracks appeared (at about 50% of the ultimate load) simultaneously in the centre of the sides in the shear spans and across the bottom surface in the centre zone. In

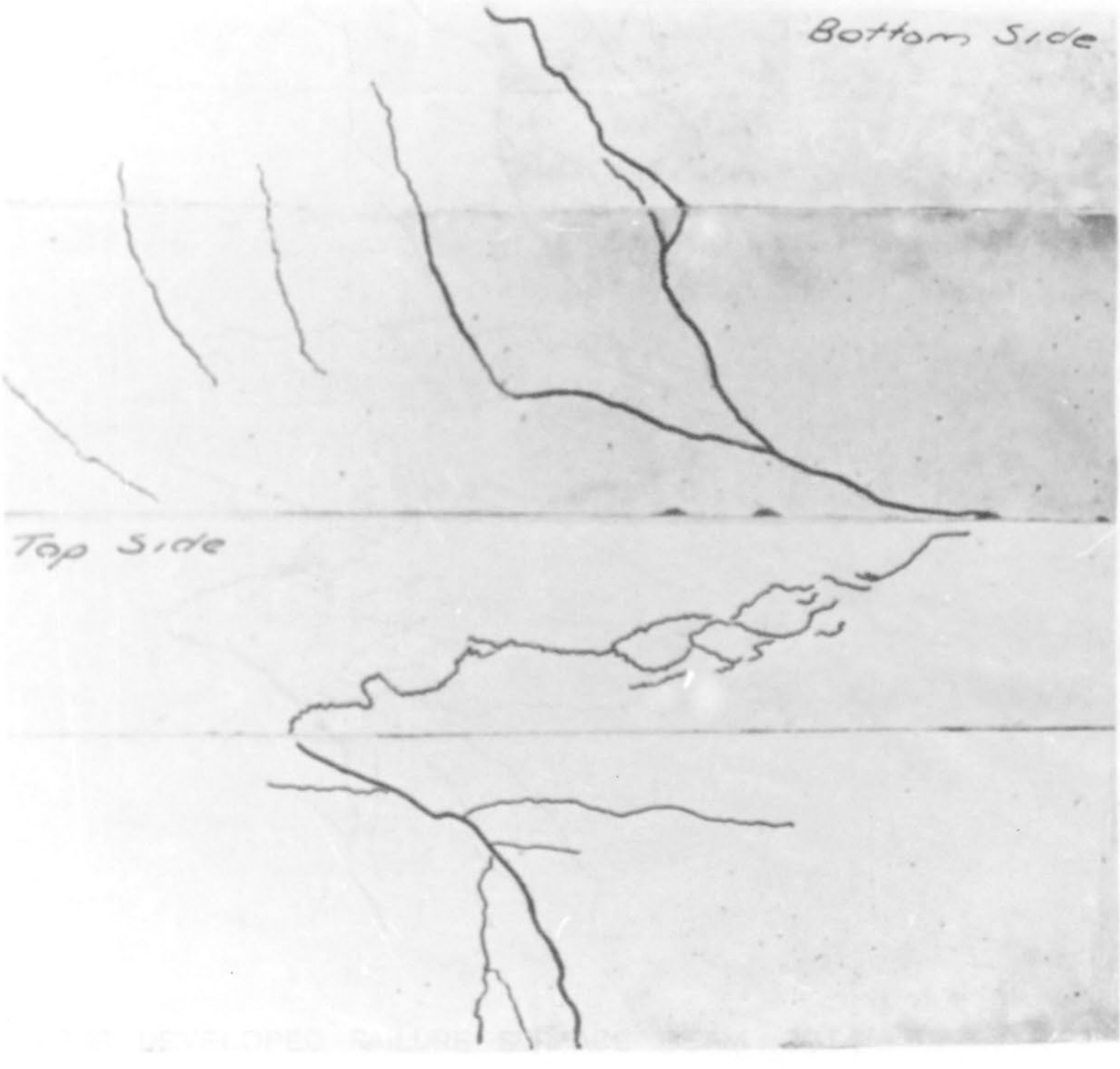


FIG. 3.15 DEVELOPED FAILURE SURFACE BEAM RU5A
MODE 1 FAILURE.

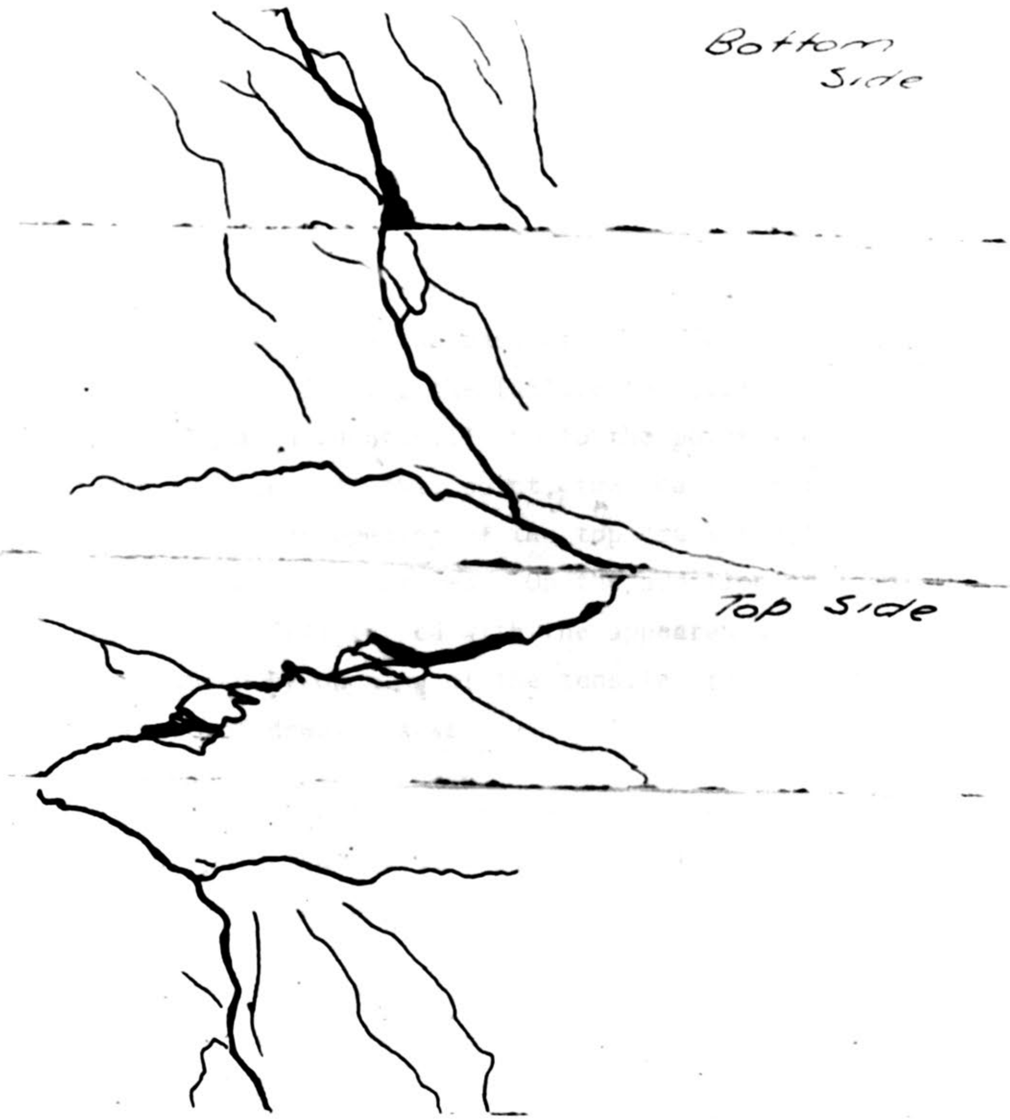


FIG. 3.16 DEVELOPED FAILURE SURFACE BEAM 36T4
MODE 1 FAILURE.

both beams, at increased loads, a crack appeared across the top surface. In the case of beam 7705 this crack occurred in the shear span and immediately opened, precipitating failure. For beam 77T5 the crack on the top face, from all appearances a cleavage crack and not associated with crushing, appeared in the central region of the total span. On the addition of further load, the crack widened and the beam rotated. Large dowel cracks then appeared and the load dropped off suddenly.

The initial behaviour of beams 77T4 and 7704 was similar to that of 77T5 and 7705. However, in the case of beam 7704, the shear spans were externally clamped forcing the failure to occur in the centre region. The two beams behaved identically up to the point when a cleavage crack crossed the top surface. At this point, the beam with the open stirrups (7704) failed with the opening of the top crack and the appearance of dowel cracks on the side surfaces. On the addition of load, the beam with the closed ties (77T4) failed with the appearance of a compression zone on the top accompanied by opening of the tensile cracks on the other faces and the appearance of dowel cracks.

Beam 7703, which was not clamped, failed in the shear span in a similar way to beam 7705. That is to say, once a cleavage crack crossed the top surface, it opened, precipitating failure.

The beams of the third series, which contained less top longitudinal steel than bottom steel, and were loaded predominantly in torsion, (beams 36T4c and 24T3) failed with the opening of tension cracks on the top and sides of the beams in a manner of a mode 3 failure. The tension cracks extended around the beams at an almost constant inclination, while compression cracks were evident on the bottom surfaces (see Figure 3.17). The beam which had the higher ratio of transverse to longitudinal steel (24T3) exhibited the more gradual failure of the two.

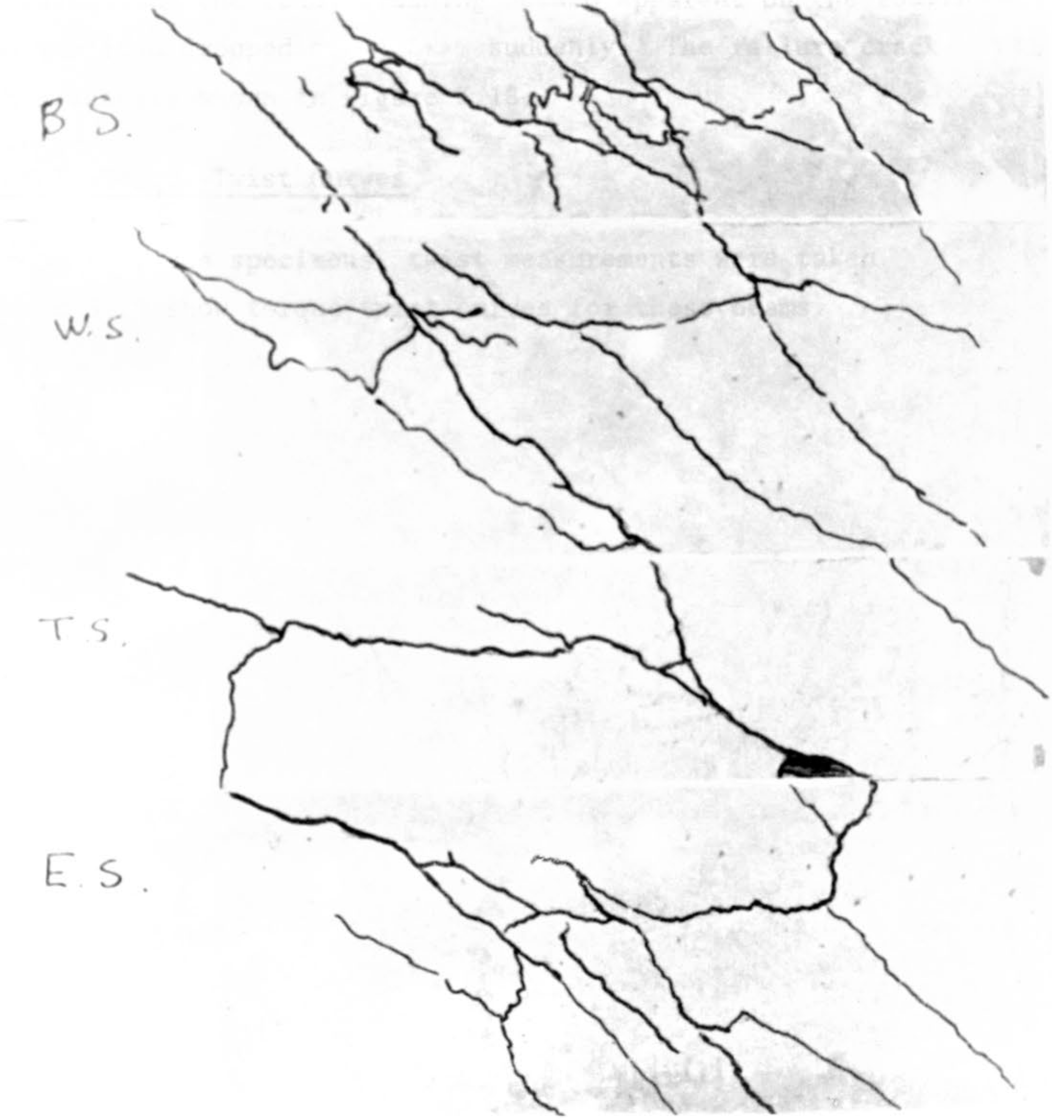


FIG. 3.17 DEVELOPED FAILURE SURFACE OF BEAM 36 T4 c
MODE 3 FAILURE

Beam 38T5 failed in the shear span in a typical mode 2 failure. The first crack was observed in the centre of the side at about 65% of the ultimate load. This crack spread as the load was increased until at 85% of the ultimate load it extended across the top, bottom and one side surface. On increasing the load, crushing became apparent on the fourth face and then the load dropped off fairly suddenly. The failure crack pattern of this beam is shown in Figure 3.18.

3.4(c) Torque-Twist Curves

For some of the specimens, twist measurements were taken. Figures 3.19 and 3.20 show torque-twist curves for these beams.

FIG. 3.18. DEVELOPED FAILURE SURFACE OF BEAM 38T5
MODE 2 SURFACE.

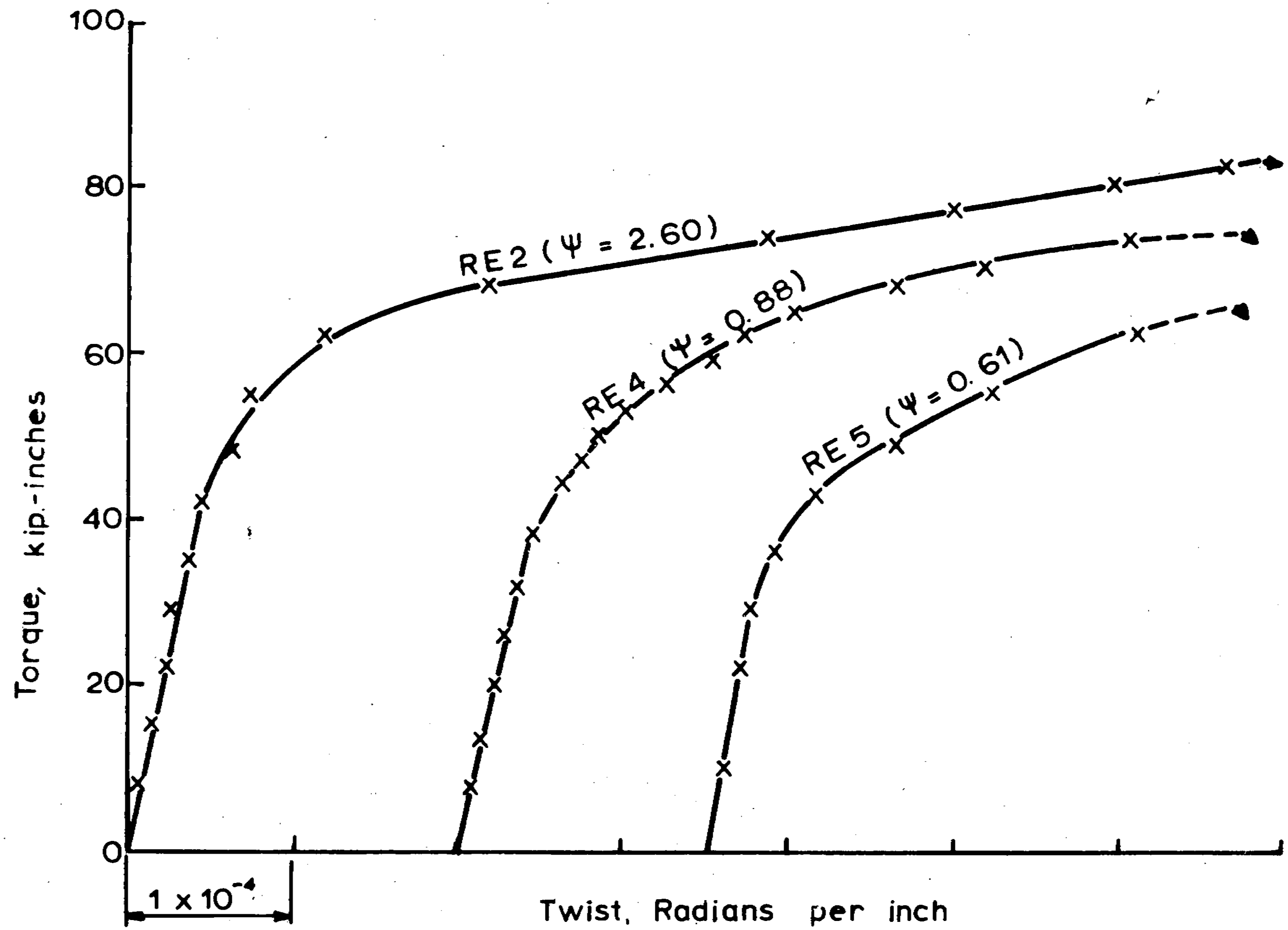


FIG. 3.19 TORQUE—TWIST CURVES FOR BEAMS CONTAINING WEB REINFORCEMENT.

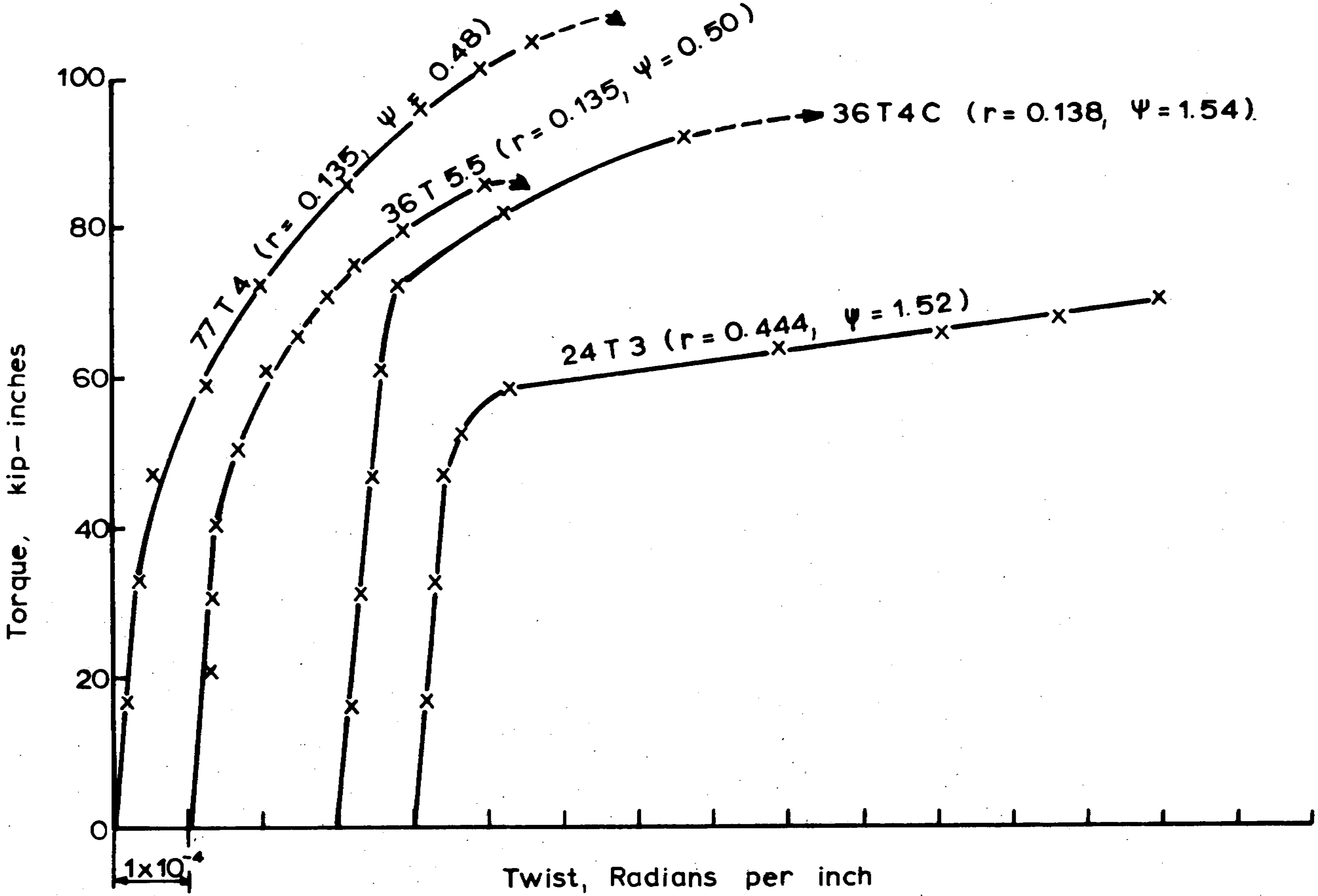


FIG. 3. 20. TORQUE—TWIST CURVES FOR BEAMS CONTAINING WEB REINFORCEMENT.

3.5 BEAMS CONTAINING LONGITUDINAL AND TRANSVERSE REINFORCEMENT - DEFORMATION SERIES

This series of tests had two aims, (i) to confirm the ultimate strength theory of design, (ii) to investigate the deformation characteristics of web reinforced beams loaded in bending and torsion. Secondary tests were carried out to verify the design method for shear and torsion.

3.5.1 Description of Specimens

The deformation series comprised twelve 10 ft. beams of 10" x 5" section. Three were tested in bending and the remainder were tested in bending and torsion. The principal variables considered were; (i) the ratio of torque to moment, and (ii) the quantity of longitudinal steel. The values of these parameters are set out in Figure 3.21.

The amount of transverse steel was computed in accordance with the design method set out in chapter 7. Thus no hoop steel was provided for pure bending and the amount of reinforcement increased with the ratio of torsion to moment. Transverse reinforcement was fabricated from ¼" diameter mild steel, flash butt welded to form closed hoops.

The main bars of the longitudinal steel were structural grade deformed bars. To obtain the required area additional small plain structural grade bars were used. The apparent values of the modulus of elasticity for both steel and concrete, were obtained from the stress strain curves given in Figure 3.22. The arrangement of the longitudinal steel is shown in Figure 3.23.

All test beams were cast from the one batch of transit mixed concrete. The strength of concrete at the time of test is given in Figure 3.24 and also in Table 3.6. Auxilliary plain concrete specimens were manufactured from the same concrete batch and were tested in pure torsion and pure bending. The results of these tests are listed in Table 3.6.

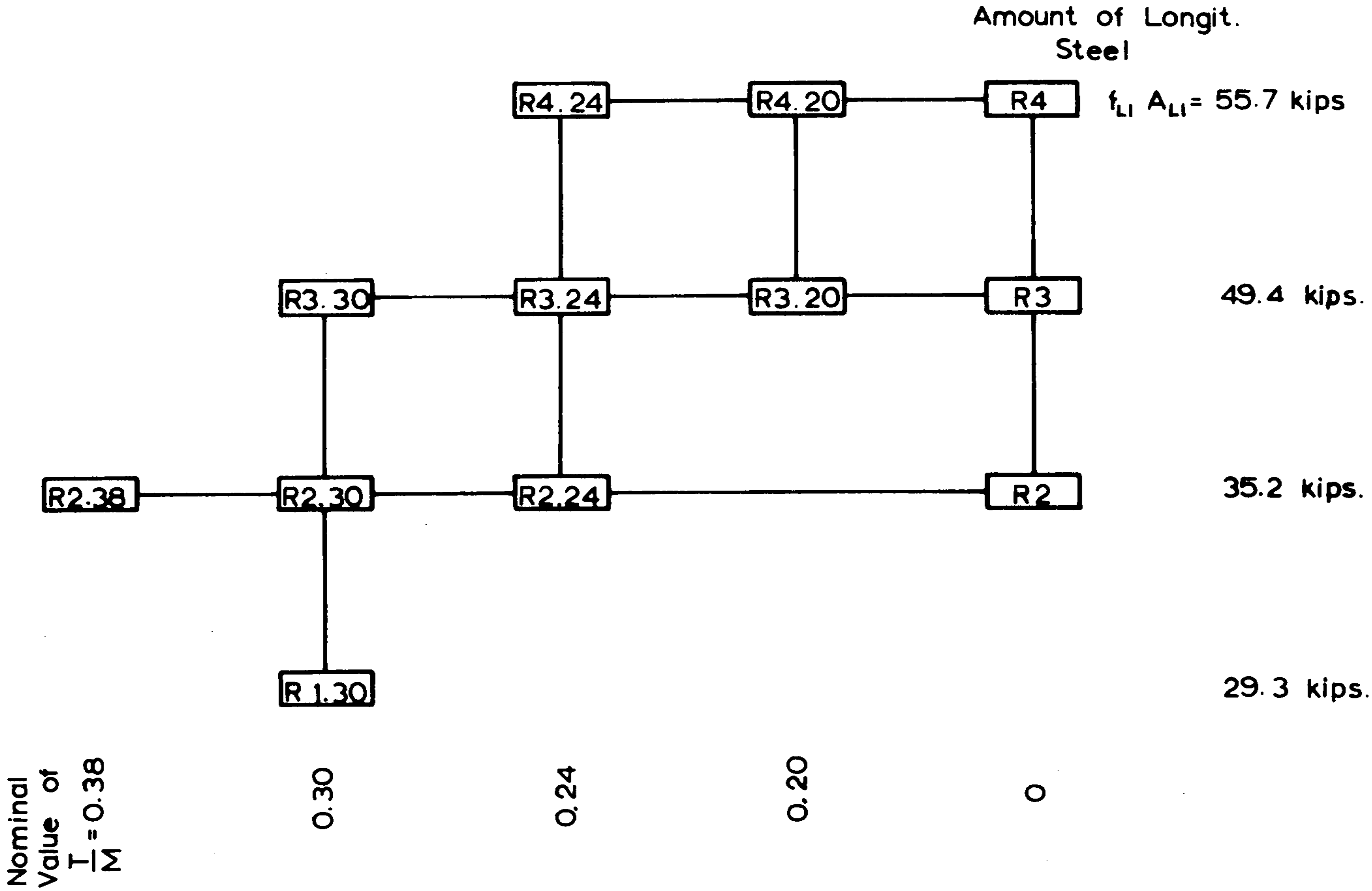


FIG.3.21 ARRANGEMENT OF PRINCIPAL VARIABLES FOR DEFORMATION SERIES

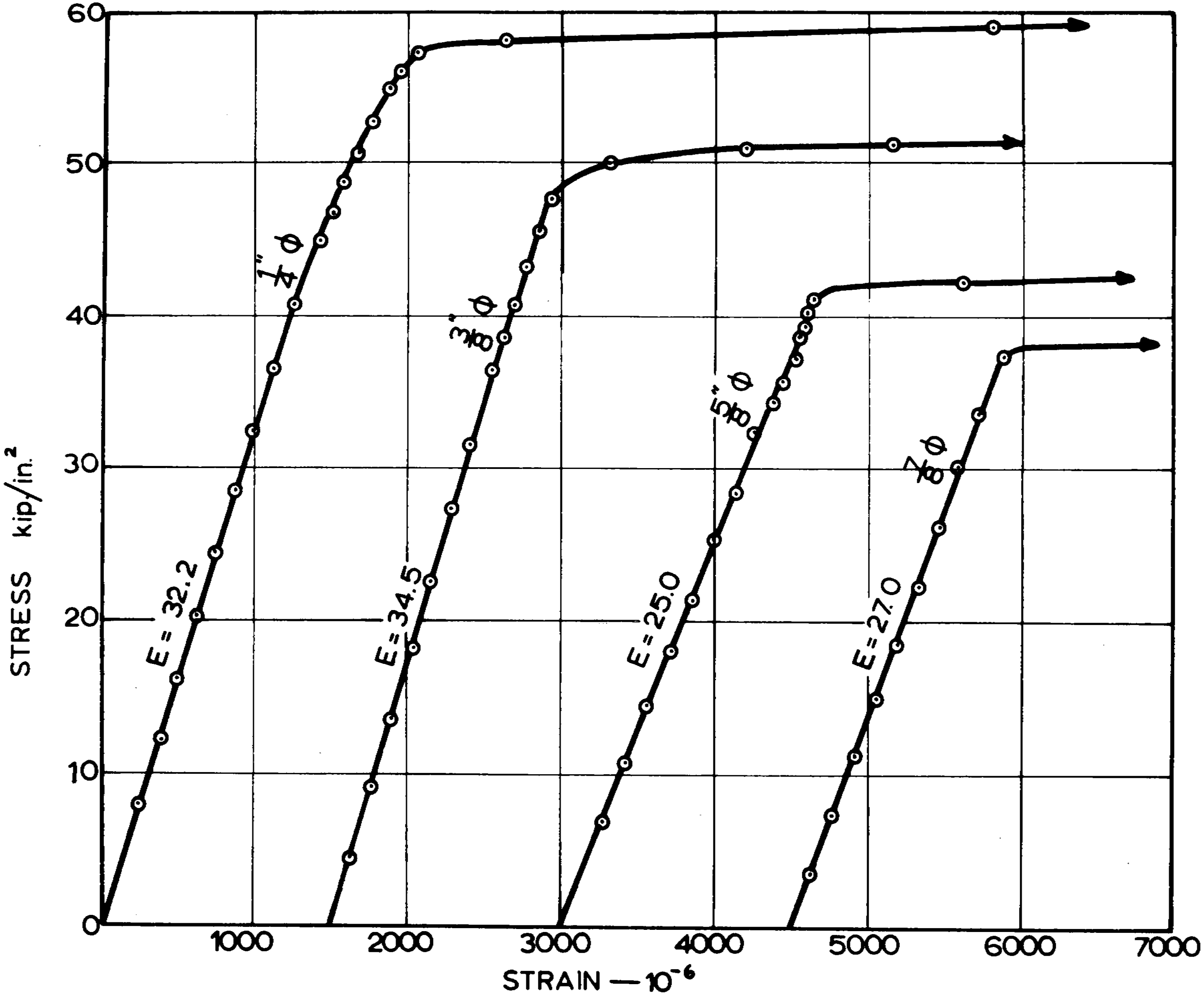


FIG. 3.22 STRESS STRAIN CURVE FOR STEEL OF THE DEFORMATION SERIES E

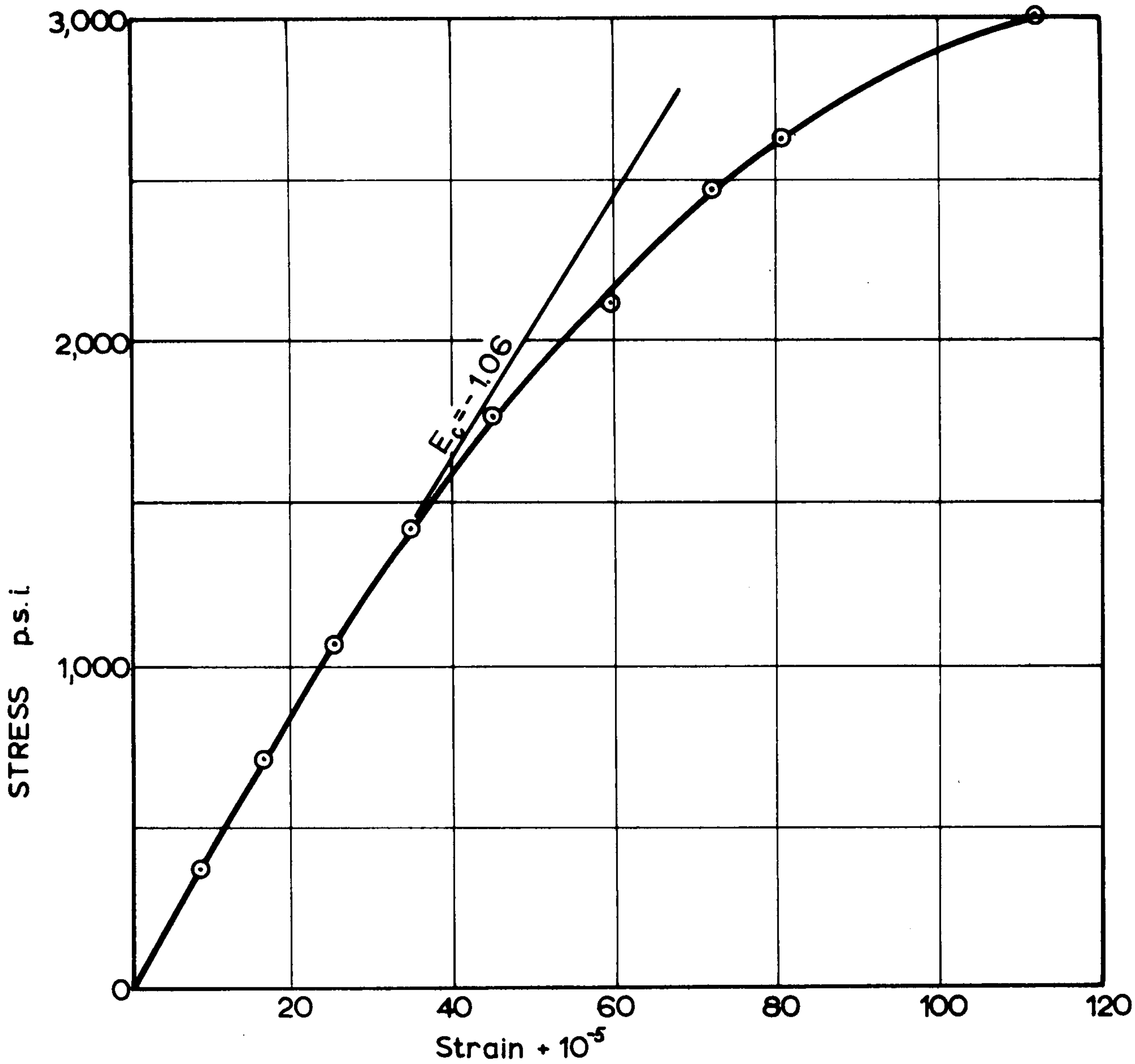


FIG. 3.22 STRESS STRAIN CURVE FOR COMPRESSION TEST ON CONCRETE OF THE DEFORMATION SERIES.

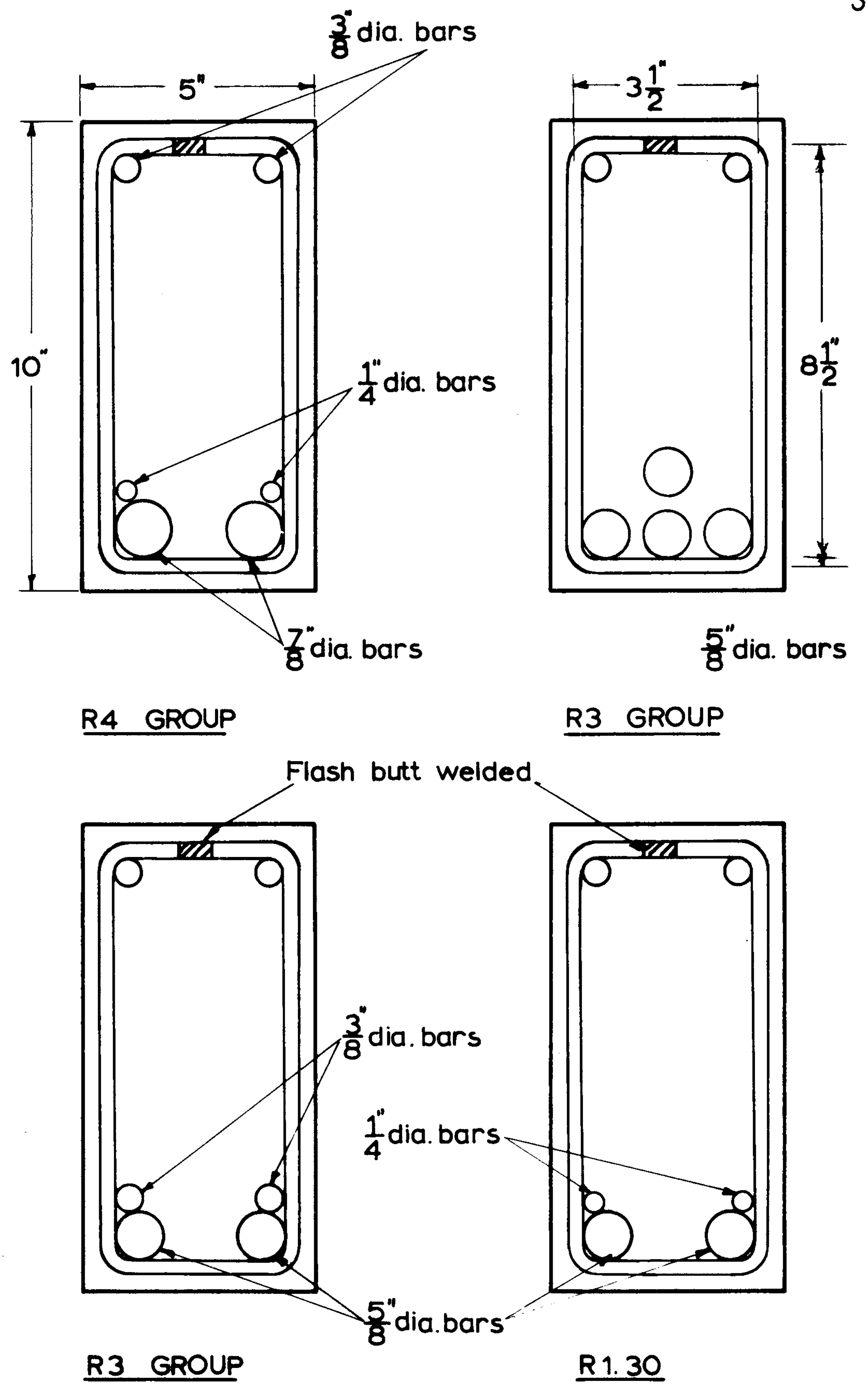


FIG. 3.23 CROSS SECTIONS OF DEFORMATION SERIES

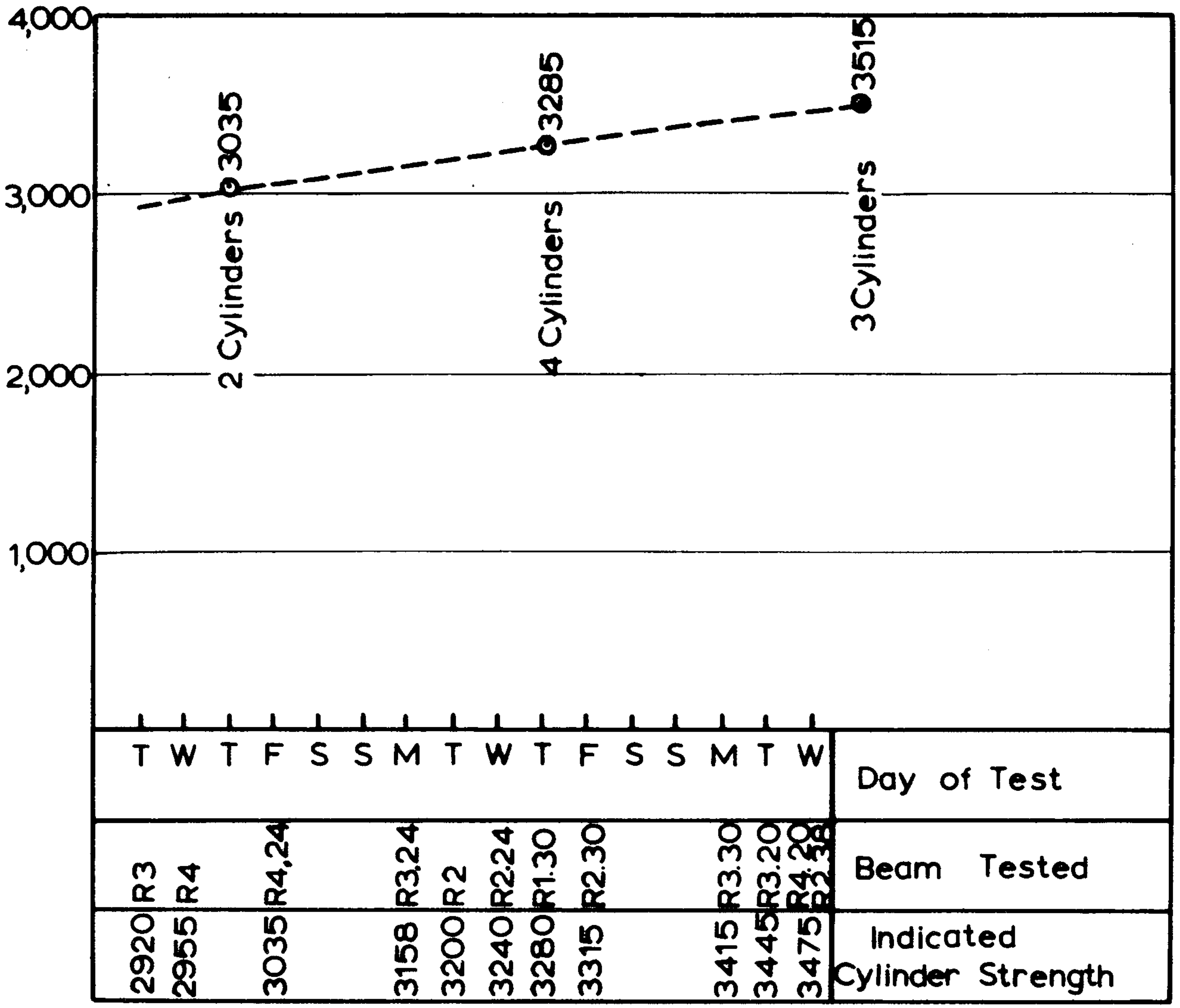


FIG.3.24 CYLINDER COMPRESSIVE STRENGTH VS DAY OF TEST.

TABLE 3.6

DETAILS OF BEAMS OF THE DEFORMATION SERIES (See also Fig. 3.23)

	Geometry (inches)				Web Steel			Longitudinal Steel				Failure Loads	
Beam No.	h	b	a_1	a_2	A_w	f_w	s	A_{L1}	f_{L1}	$\frac{A_{L3} f_{L3}}{A_{L1} f_{L1}}$	f'_c p.s.i.	T kip. in.	M kip. in.
R4.20	10.0	5.0	1.5	1.2	0.049	58.6	2.59	1.420	40.6	0.155	3474	59.9	331.0
R4.24	10.0	5.0	1.5	1.2	0.049	58.6	2.25	1.420	40.6	0.155	3034	56.5	264.0
R3.20	10.0	5.0	1.5	1.1	0.049	58.6	3.12	1.220	40.6	0.180	3444	50.7	252.0
R3.24	10.0	5.0	1.5	1.1	0.049	58.6	2.72	1.220	40.6	0.180	3157	53.7	230.0
R3.30	10.0	5.0	1.5	1.1	0.049	58.6	2.38	1.220	40.6	0.180	3414	61.6	207.0
R2.24	10.0	5.0	1.7	1.2	0.049	58.6	3.80	0.830	42.4	0.265	3239	44.2	205.0
R2.30	10.0	5.0	1.7	1.2	0.049	58.6	3.32	0.830	42.4	0.265	3314	49.7	176.0
R2.38	10.0	5.0	1.7	1.2	0.049	58.6	2.84	0.830	42.4	0.265	3474	53.4	138.0
R1.30	10.0	5.0	1.2	1.2	0.049	58.6	3.95	0.710	41.3	0.310	3279	41.8	146.0
RP1	10.3	5.2	(Plain Concrete Control Tests)								3460	35.6	0
RP2	10.0	5.0									3076	0	43.1

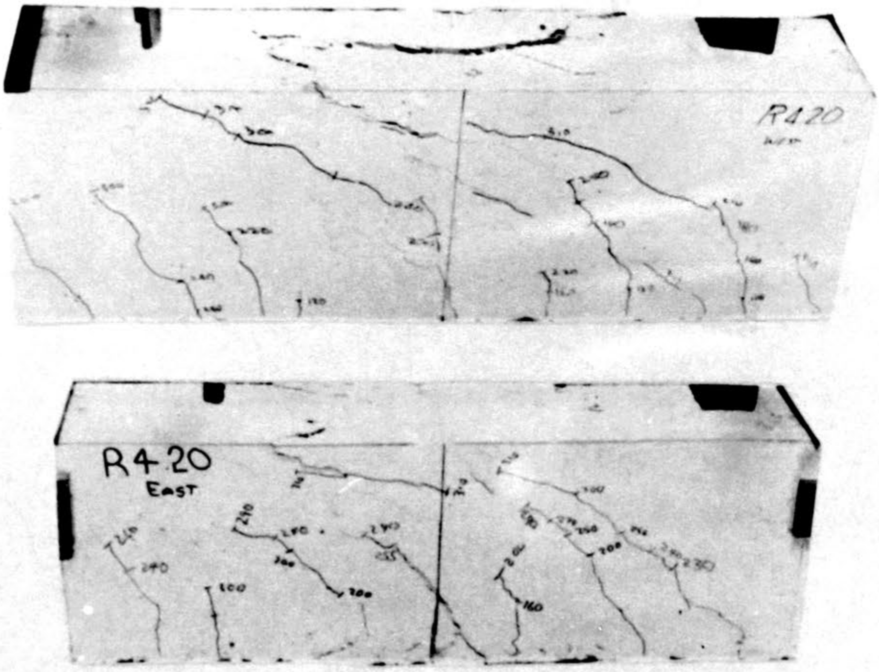
The main bending and torsion tests were carried out using the two point loading rig described in a previous section (3.1). In addition to the main tests on the centre sections of the beam secondary tests were carried out on the shear spans. For the secondary tests the one point loading system (see 3.1) was used. Details of the secondary tests are contained in Table 3.7. For some beams semi bright ¼" diameter steel was used in the stirrups. This steel had a "yield point" of 73.6 kips/in^2.

3.5.1 Test Behaviour

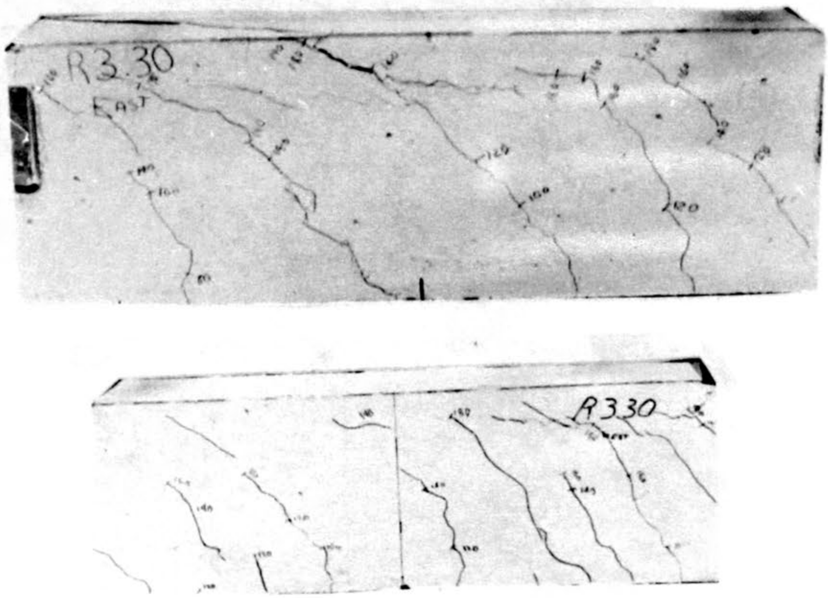
Beam R4 which was tested in simple flexure failed by crushing of the concrete prior to yield of the longitudinal steel. Tensile cracks did not open at failure and the depth of crushing of the concrete was 5" to 6". Beams R2 and R3, which were more lightly reinforced, failed in the more usual manner with gradual opening of the tension cracks before compression failure of the concrete.

The specimens subjected to bending and torsion all behaved in a similar manner. Cracks spread from the bottom of the beam gradually becoming more inclined to the vertical. The height of the cracks for the bending and torsion specimens was considerably greater than it was for the corresponding pure bending specimen at the same moment. The ultimate load was reached when a tensile crack opened and the concrete on the top surface failed. For the beams of this series, the compression zone failed with splitting of the top surface. The general appearance at failure of typical beams of this series is shown in the photographs of Figure 3.25.

In beam R4.24 a premature failure in the shear span prevented the full ultimate load in the centre test length from being reached. Examination of the beam and comparison with subsequent tests indicated that a bending and torsion failure was imminent at the maximum load

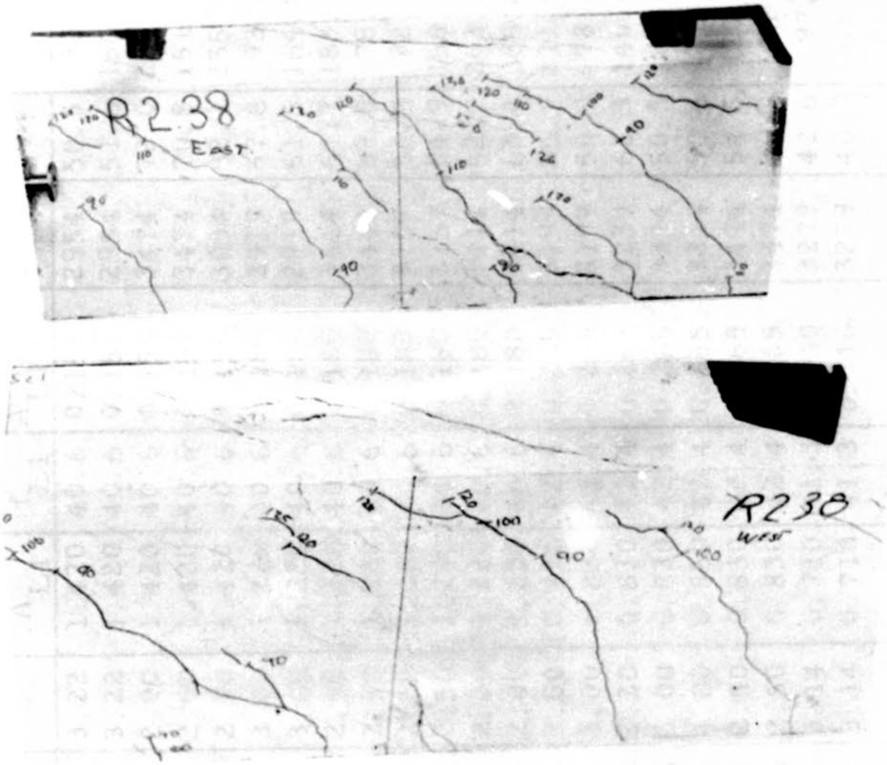


R4.20

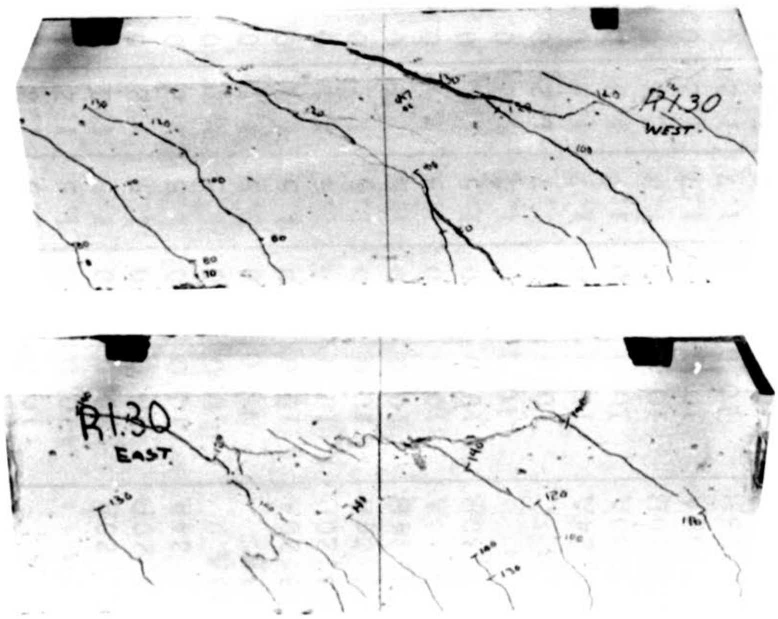


R3.30

FIG. 3.25 FAILURE SURFACES OF BEAMS OF THE DEFORMATION SERIES.



R2.38



R1. 30

FIG. 3.25 FAILURE SURFACES OF BEAMS OF THE DEFORMATION SERIES

TABLE 3.7

DETAILS OF SECONDARY TESTS ON BEAMS OF THE DEFORMATION SERIES

	Geometry (inches)				Web Steel			Longitudinal Steel				Failure Loads		
Beam No.	h	b	a_1	a_2	A_w	f_w	s	A_{L1}	f_{L1}	$\frac{A_{L3}f_{L3}}{A_{L1}f_{L1}}$	f'_c p.s.i.	T kip. in.	M kip. in.	V kips.
R4A	10.0	5.0	1.5	1.2	0.049	58.6	3.22	1.420	40.6	0.155	2954	50.5	92.1	3.94
R4B	10.0	5.0	1.5	1.2	0.049	58.6	3.22	1.420	40.6	0.155	2954	51.6	166.0	6.92
R4.20A	10.0	5.0	1.5	1.2	0.049	73.6	2.00	1.420	40.6	0.155	3474	72.0	220.0	9.08
R4.20B	10.0	5.0	1.5	1.2	0.049	73.6	2.00	1.420	40.6	0.155	3474	70.8	151.0	6.30
R4.24A	10.0	5.0	1.5	1.2	0.049	58.6	2.00	1.420	40.6	0.155	3034	62.1	255.0	8.21
R3A	10.0	5.0	1.5	1.1	0.049	73.6	3.00	1.220	40.6	0.180	2919	57.6	92.1	3.94
R3B	10.0	5.0	1.5	1.1	0.049	73.6	3.00	1.220	40.6	0.180	2919	51.5	166.0	6.92
R3.20A	10.0	5.0	1.5	1.1	0.049	58.6	2.50	1.220	40.6	0.180	3444	61.7	183.0	7.60
R3.20B	10.0	5.0	1.5	1.1	0.049	58.6	2.50	1.220	40.6	0.180	3444	59.0	78.9	3.41
R3.24A	10.0	5.0	1.5	1.1	0.049	73.6	2.50	1.220	40.6	0.180	3157	56.5	92.9	3.97
R3.24B	10.0	5.0	1.5	1.1	0.049	73.6	2.50	1.220	40.6	0.180	3157	54.9	158.0	6.57
R3.30A	10.0	5.0	1.5	1.1	0.049	73.6	2.00	1.220	40.6	0.180	3414	62.2	203.0	8.39
R3.30B	10.0	5.0	1.5	1.1	0.049	73.6	2.00	1.220	40.6	0.180	3414	63.4	86.8	3.73
R2A	10.0	5.0	1.7	1.2	0.049	58.6	3.00	0.830	42.4	0.265	3199	55.7	111.0	4.71
R2B	10.0	5.0	1.7	1.2	0.049	58.6	3.00	0.830	42.4	0.265	3199	50.9	78.7	3.42
R2.24A	10.0	5.0	1.7	1.2	0.049	58.6	3.30	0.830	42.4	0.265	3239	62.3	140.0	5.85
R2.30A	10.0	5.0	1.7	1.2	0.049	73.6	3.00	0.830	42.4	0.265	3314	50.3	104.0	4.43
R2.30B	10.0	5.0	1.7	1.2	0.049	73.6	3.00	0.830	42.4	0.265	3314	50.6	71.8	3.13
R2.38A	10.0	5.0	1.7	1.2	0.049	73.6	2.50	0.830	42.4	0.265	3474	54.6	120.0	5.08
R2.38B	10.0	5.0	1.7	1.2	0.049	73.6	2.50	0.830	42.4	0.265	3474	48.8	70.7	3.08
R1.30A	10.0	5.0	1.2	1.2	0.049	58.6	3.14	0.710	41.3	0.310	3279	42.6	97.1	4.13
R1.30B	10.0	5.0	1.2	1.2	0.049	58.6	3.14	0.710	41.3	0.310	3279	40.9	61.8	2.73

applied. To avert shear span failures, clamps were placed on the shear span during subsequent tests.

After the primary tests were completed the shear spans of each beam were reloaded in bending, torsion and shear. Although the shear spans were slightly cracked prior to retesting, tests on other beams (see 3.4.3) and tests by Collins indicated that the ultimate strength would not be appreciably affected by these cracks. Extensive cracking in the shear span during the main tests was of course prevented by the external clamps.

A general aim of the secondary tests was to test one end at a ratio of loads that should produce a mode 3 failure (see 3.4), and the other end at a somewhat higher ratio of shear to torsion, to induce a mode 2 failure. Beams for which mode 2 failures were predicted did in fact behave in a manner similar to the idealized mode. This type of behaviour is illustrated in Figure 3.26. Specimens for which mode 3 failures were predicted definitely failed with opening of cracks on the sides and top of the beam. A photograph of a beam that has failed in this mode is given in Figure 3.27.

3.5.3 Deformation Measurements

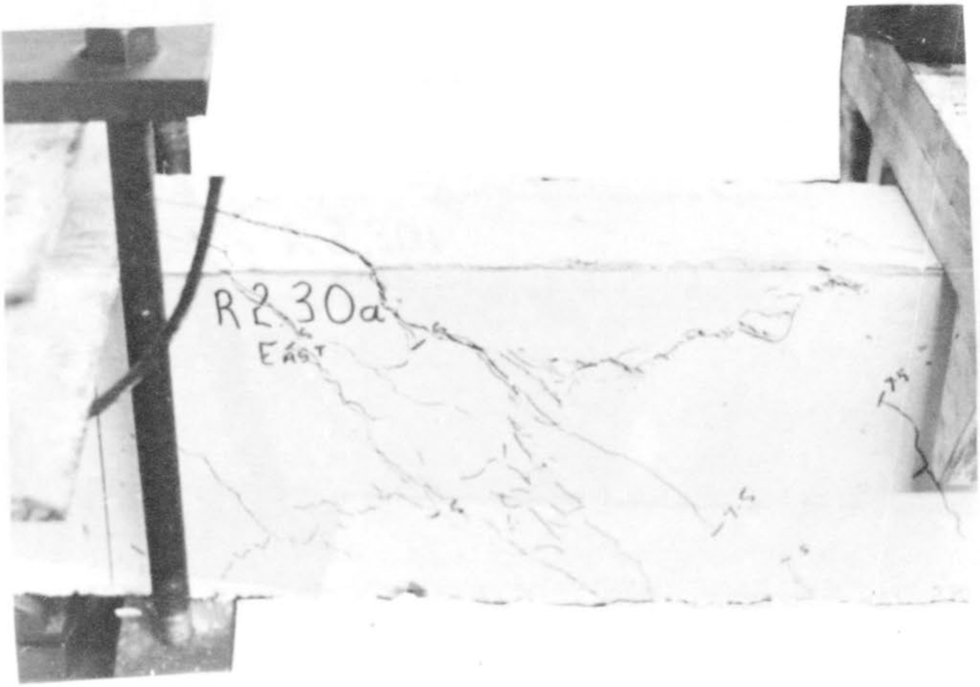
For the deformation series of beams the following measurements were taken:

(i) Deflections

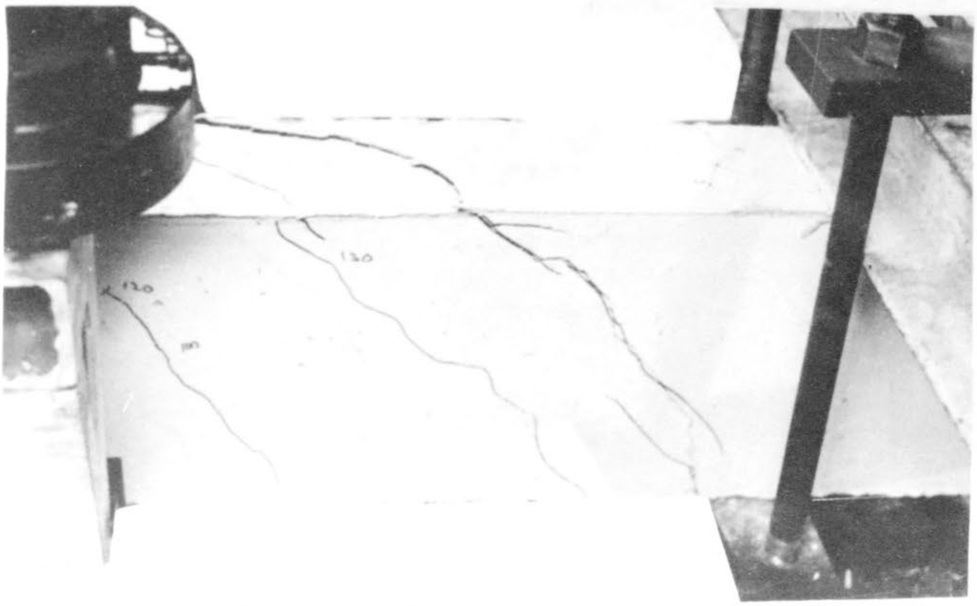
(ii) Rotations

and (iii) Strain in the longitudinal and transverse steel.

In this section only the experimental phenomena will be reported. The theoretical treatment of deformations will be given in Chapter 8.

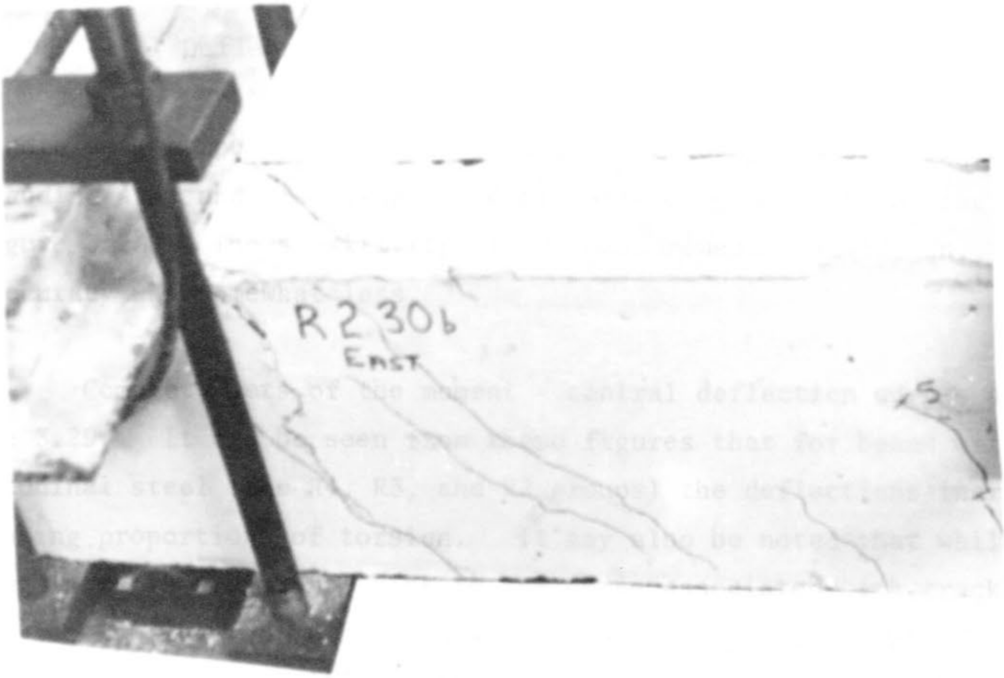


Compression Face



Tension Face

FIG. 3. 26 FAILURE SURFACE OF BEAN R2.30a
PREDICTED MODE 2 FAILURE



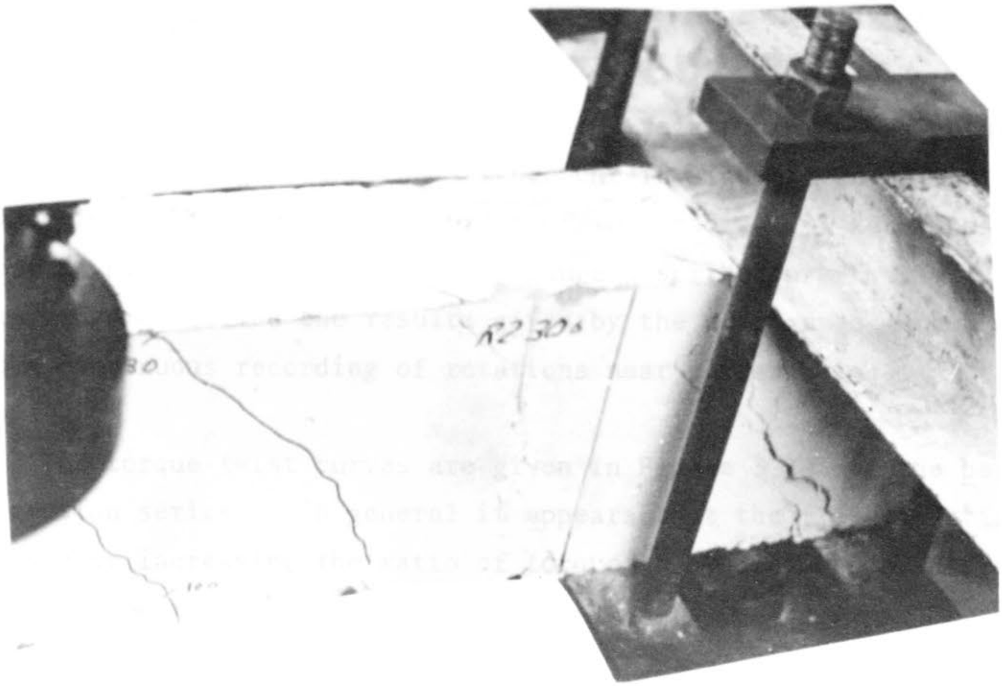


FIG 3 27 FAILURE SURFACE OF BEAM R2 30b
PREDICTED MODE 3 FAILURE

(i) Deflections

Both deflections and rotations were recorded by means of rigid outstanding arms and dial gauges. This arrangement is shown diagrammatically in Figure 3.28. The sensitivity of the measurements was 10^{-5} in., although the accuracy was somewhat less.

Complete sets of the moment - central deflection curves are given in Figure 3.29. It can be seen from these figures that for beams with the same longitudinal steel (the R4, R3, and R2 groups) the deflections increase with increasing proportions of torsion. It may also be noted that while in pure bending there is a distinct kink in the curve associated with cracking, the curves for bending and torsion show a gradual change of slope. The stages at which clamps were applied to the shear spans are marked on the curves. the presence of clamps does not seem to have affected the test results.

(ii) Rotations

The main part of the torque-rotation relationships was obtained from dial gauge readings using the arrangement in Figure 3.28. The accuracy of this method was of the order of 10^{-5} radians. Rotations were also recorded by the special inductance gauge described previously. Reading from this gauge confirmed the results given by the dial gauge method and provided a continuous recording of rotations near ultimate load.

The torque-twist curves are given in Figure 3.30 for the beams of the deformation series. In general it appears that the flexural stiffness is decreased by increasing the ratio of torque to bending and by decreasing the amount of longitudinal steel.

(iii) Strain in the longitudinal and transverse steel.

Electric resistance strain gauges were attached to the reinforcing

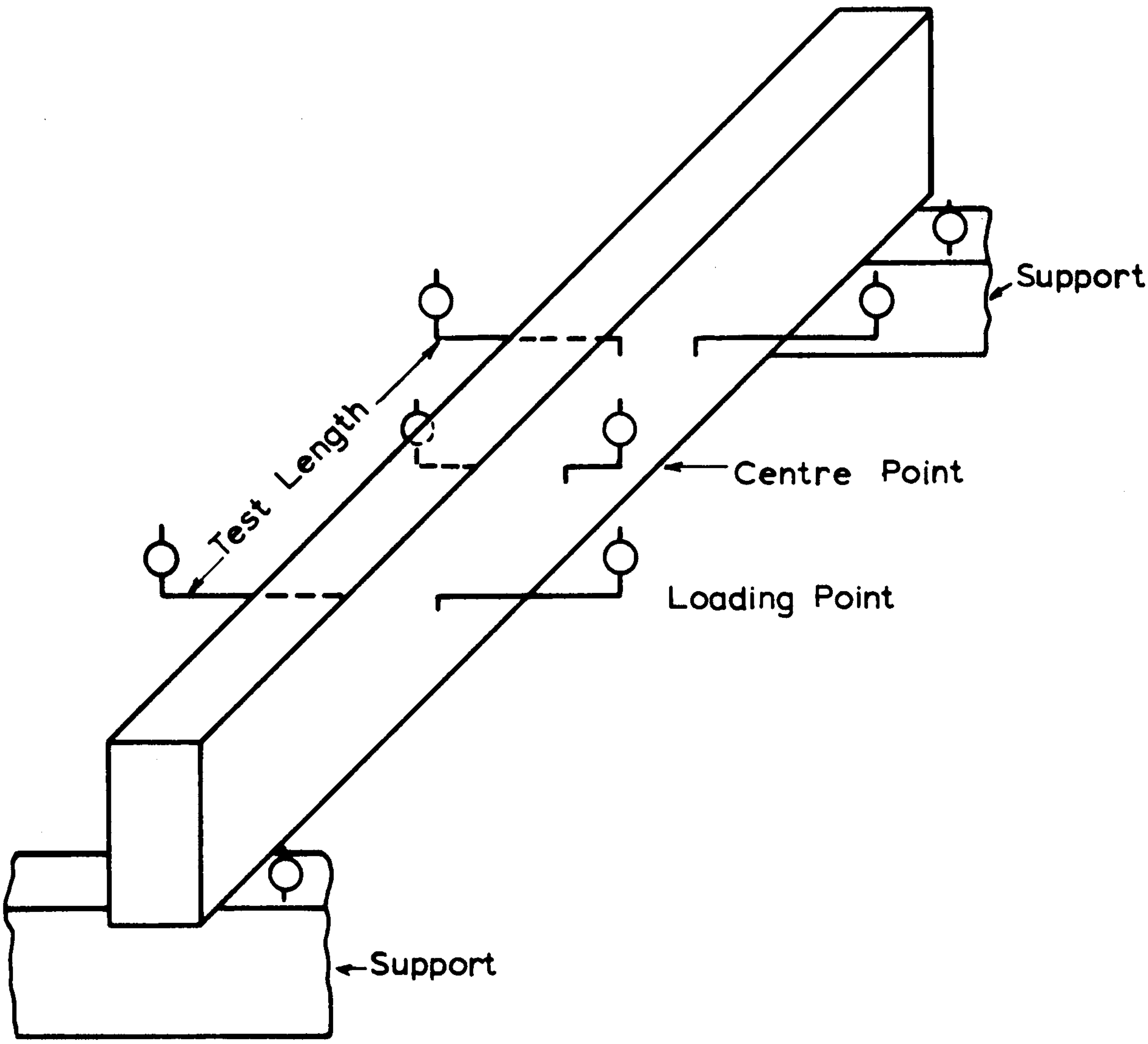


FIG. 3.28 POSITION OF DIAL GAUGES

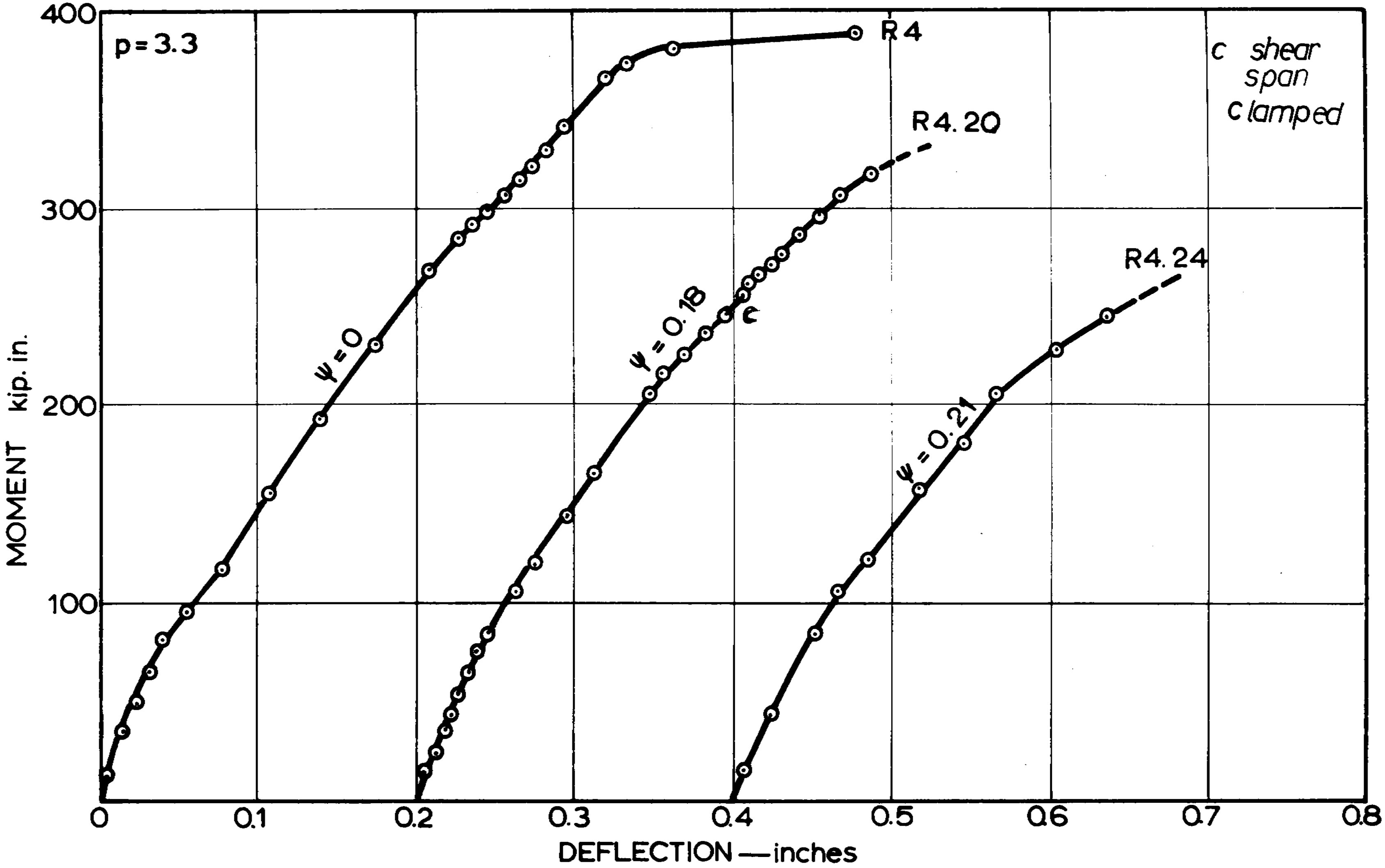


FIG. 3.29 MOMENT CENTRAL DEFLECTION FOR R4 GROUP.

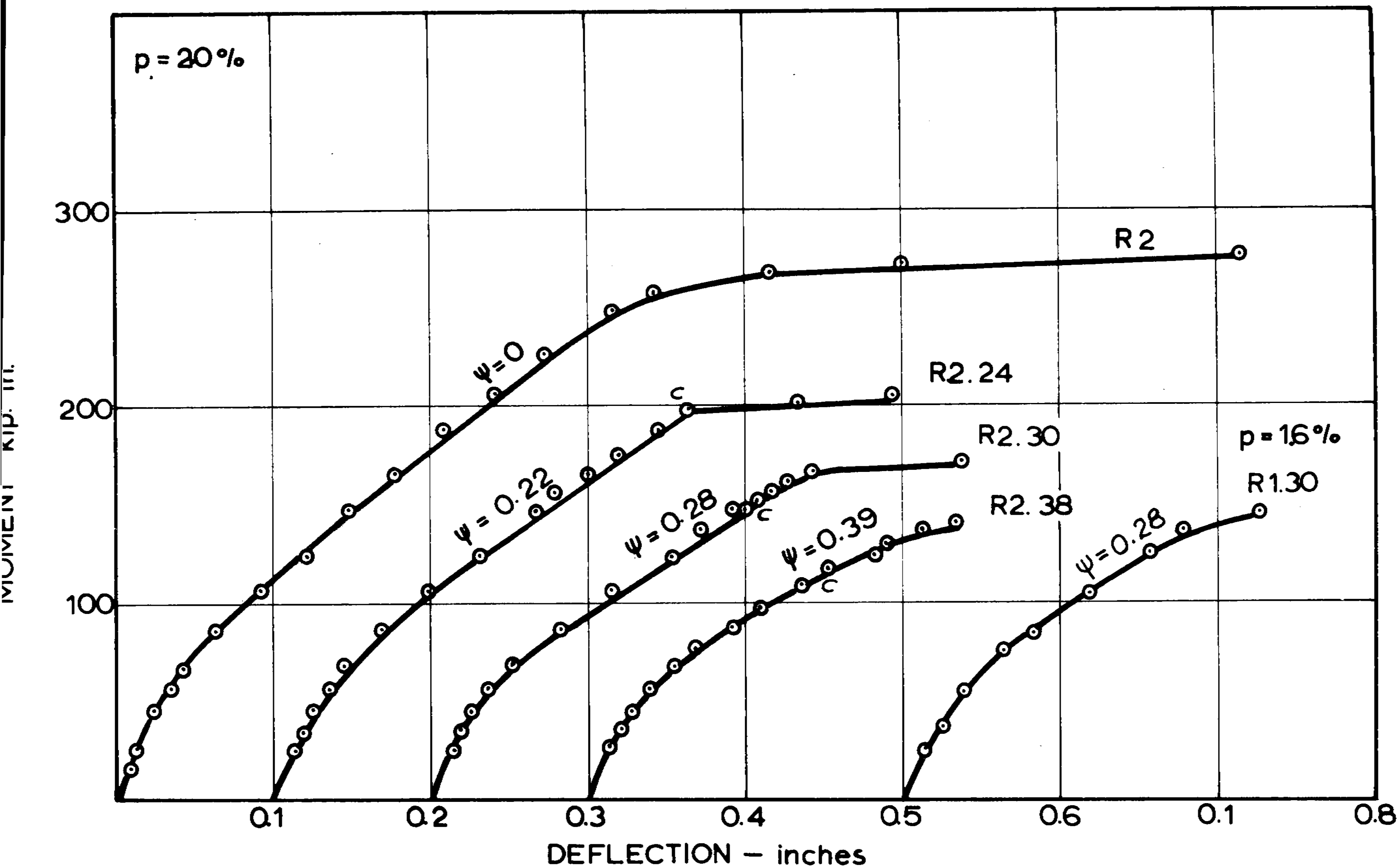


FIG. 3.29 MOMENT CENTRAL DEFLECTION FOR R2 GROUP AND BEAM R 1.30

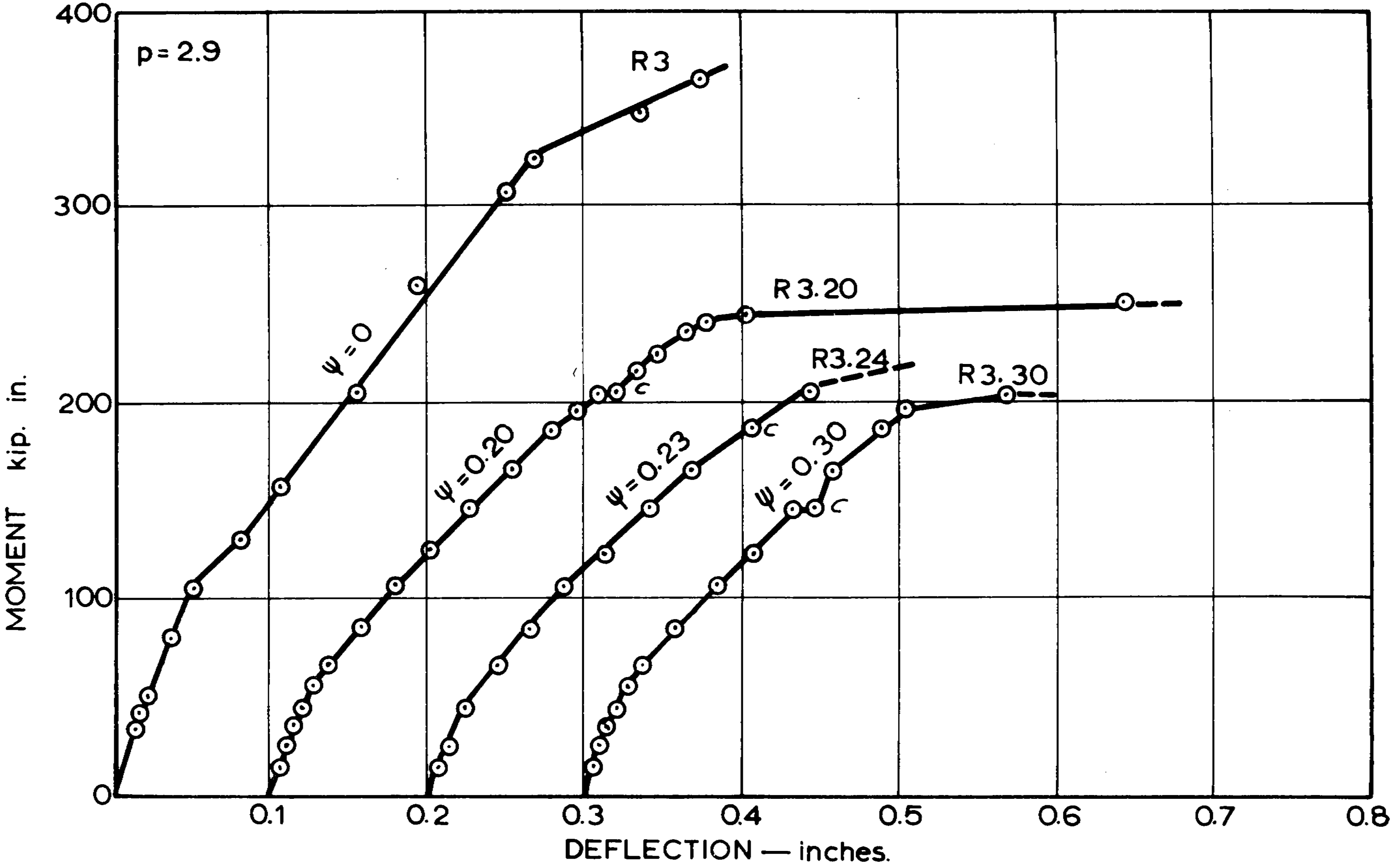


FIG. 3.29 MOMENT — CENTRAL DEFLECTION FOR GROUP R3

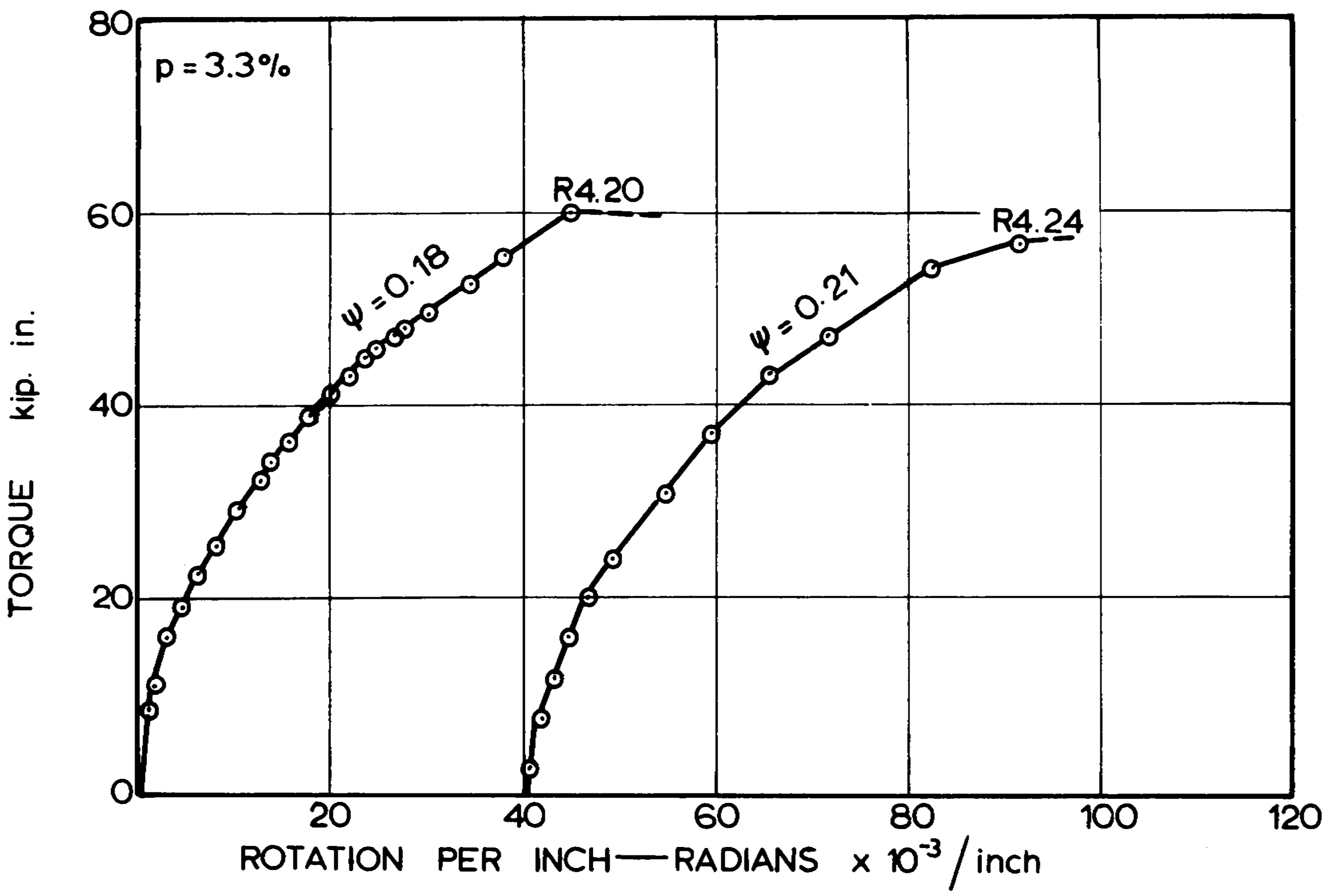


FIG. 3.30 TORQUE-TWIST RELATIONSHIP FOR R4 GROUP

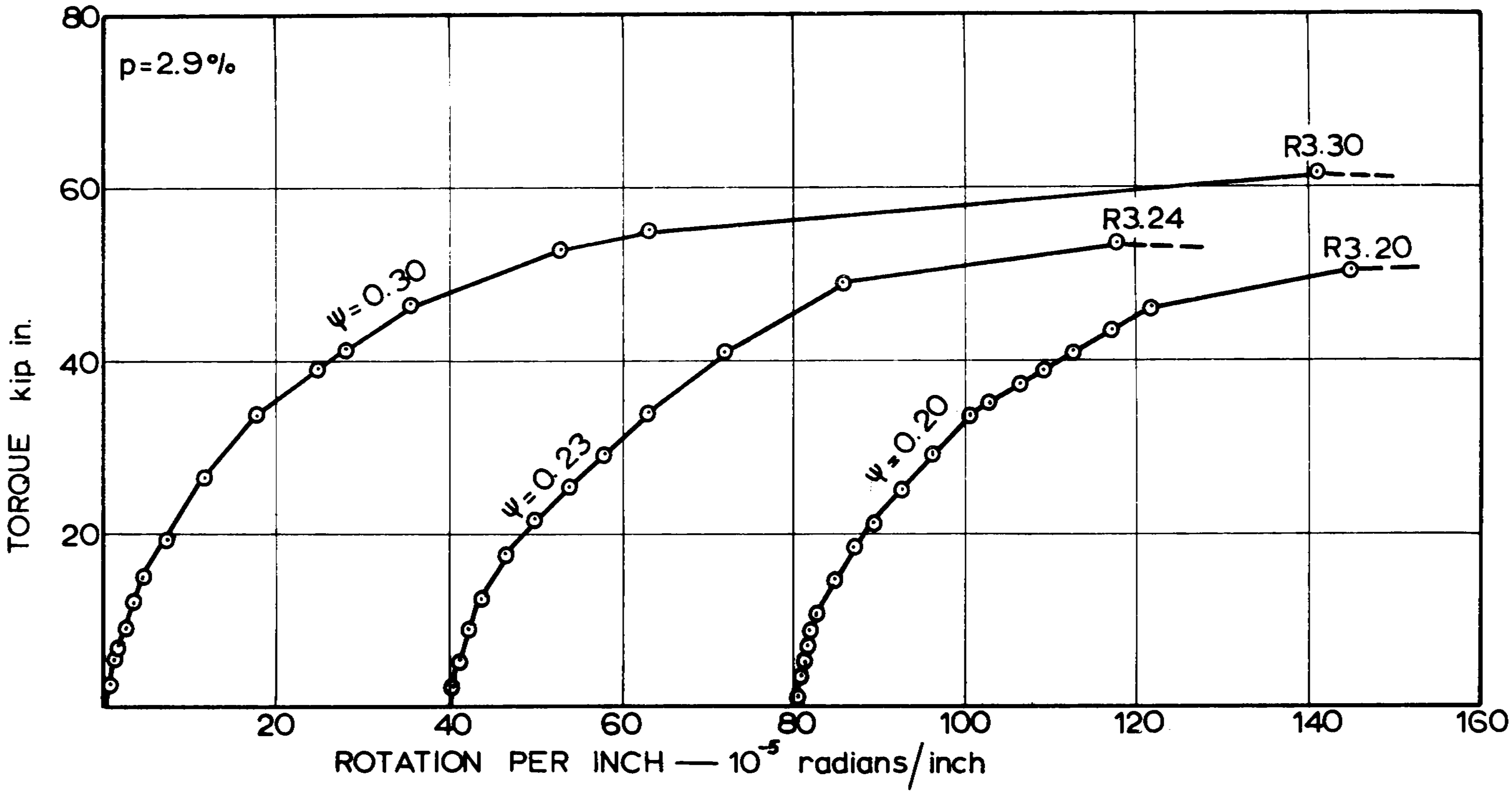


FIG.3. 30 TORQUE TWIST RELATIONSHIP FOR R 3 GROUP

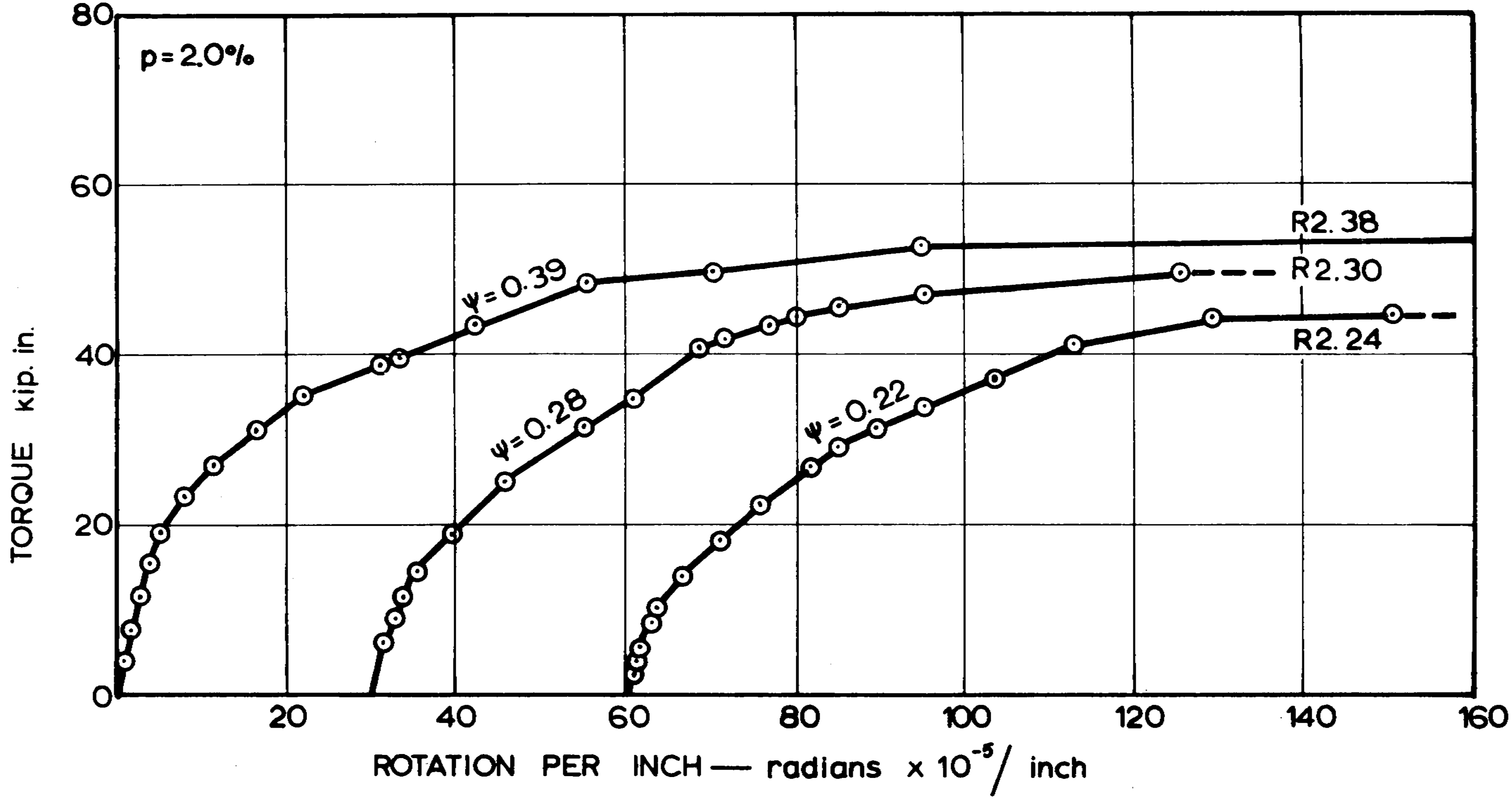


FIG. 3.30 TORQUE TWIST RELATIONSHIP FOR R2 GROUP

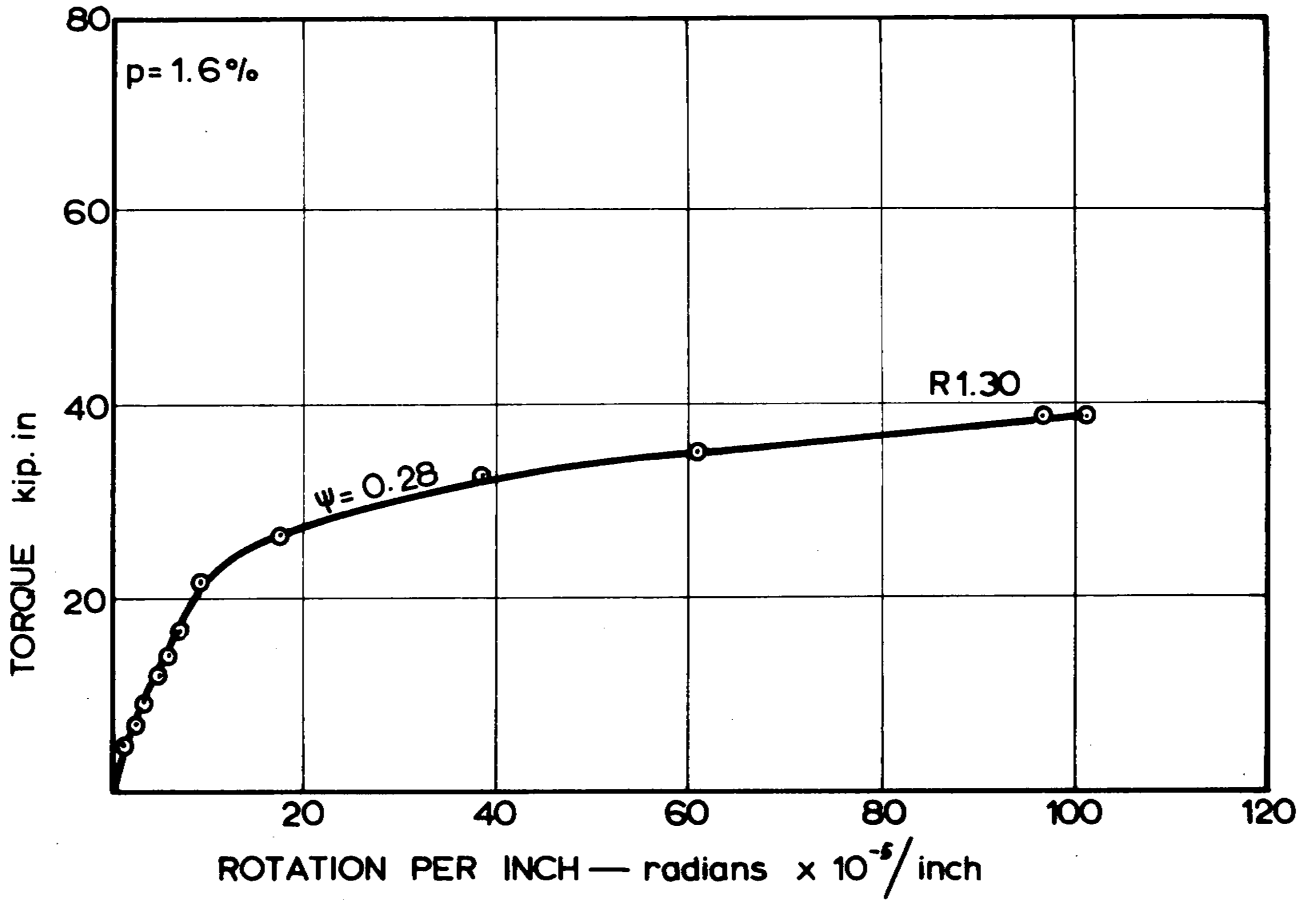


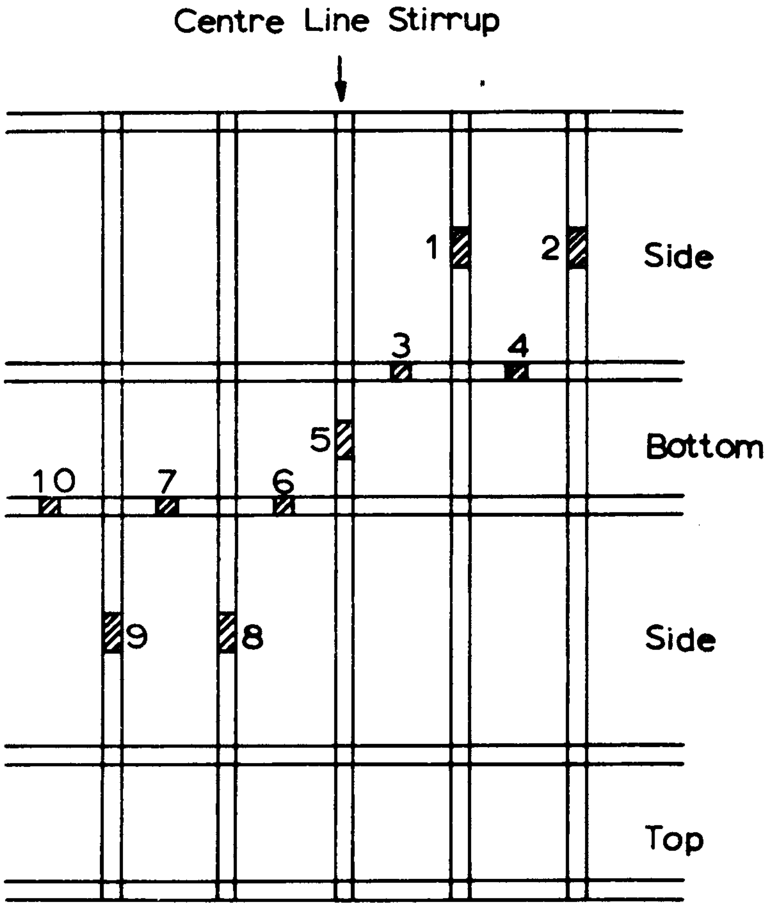
FIG. 3.30 TORQUE TWIST RELATIONSHIP FOR BEAM R1. 30

steel prior to casting. The position of the gauges is shown in Figure 3.31. Waterproofing of the gauges was achieved by layers of P.V.C. insulating tape, epoxy resin and P.V.C. insulating tape.

The relationships between the applied bending moment and the longitudinal steel strain are shown in Figure 3.32. This figure demonstrates that torsion appreciably increases the strain in the longitudinal steel.

Restrictions on space prevented efficient waterproofing of gauges attached to the transverse legs of the hoops. The results of strains in the vertical legs of the hoops are presented in Figure 3.33. The torque-steel strains relationships follow the same trends as the torque-twist curves.

The maximum strains recorded were frequently below yield strains. In part this must be attributed to the fact that the failure crack did not cross the steel at the position of the gauges.



DEVELOPED VIEW OF REINFORCING CAGE

BEAM	GAUGE POSITIONS
R4	3, 6
R4.20	6
R4.24	2,6,9
R3	3
R3.20	2,6,9
R3. 24	2, 3,9,10
R3. 30	2,3,6,9
R2	3
R2.24	1,6,8
R2.30	2,3,6,9
R2. 38	2,9,6,3
R1. 30	1,3,8

FIG.3.31 POSITIONS OF STRAIN GAUGES FOR BEAMS OF DEFORMATION SERIES

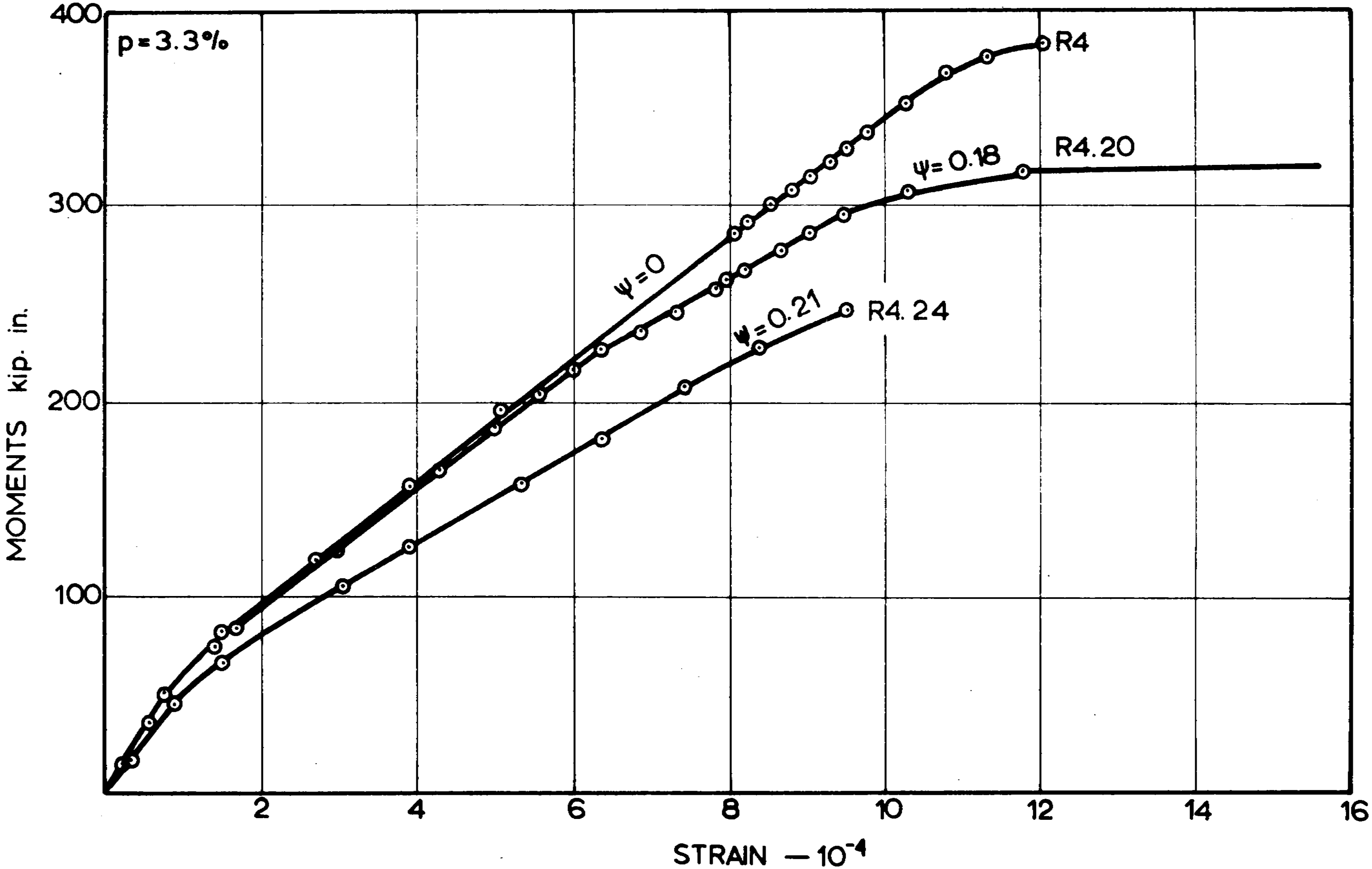


FIG. 3.32 LONGITUDINAL STEEL STRAIN FOR R4 GROUP

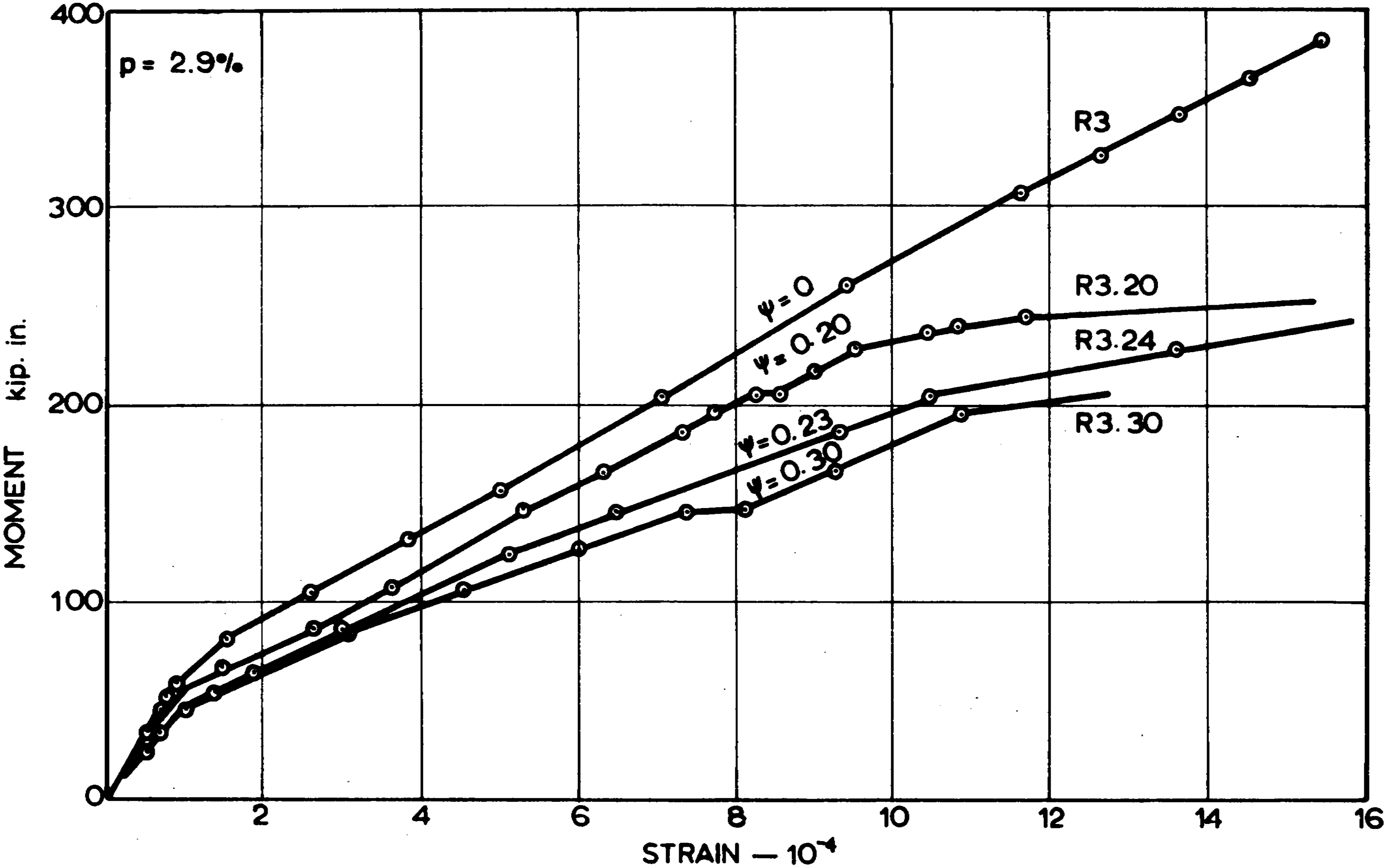


FIG. 3.32 LONGITUDINAL STEEL STRAIN FOR R3 GROUP

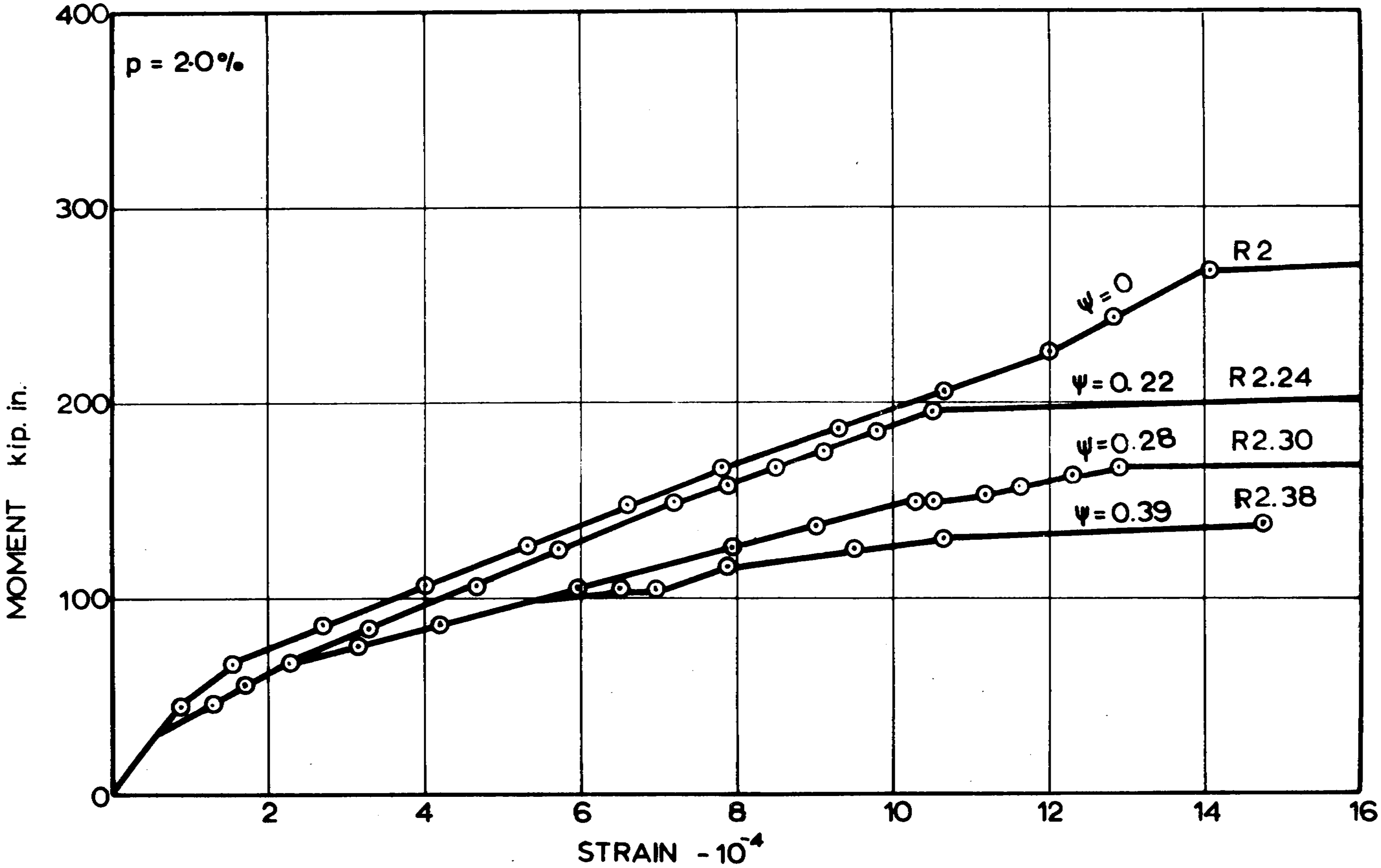


FIG. 3.32 LONGITUDINAL STEEL STRAIN FOR R 2 GROUP.

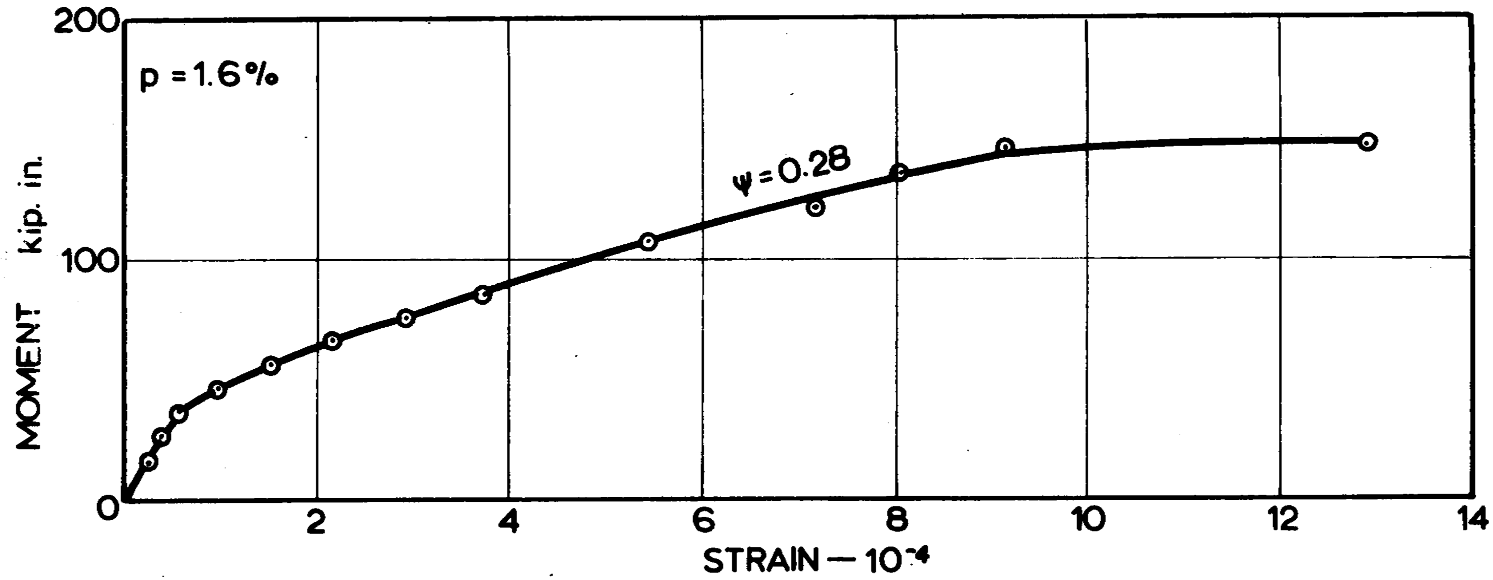


FIG. 3.32 LONGITUDINAL STEEL STRAIN FOR BEAM 1.30

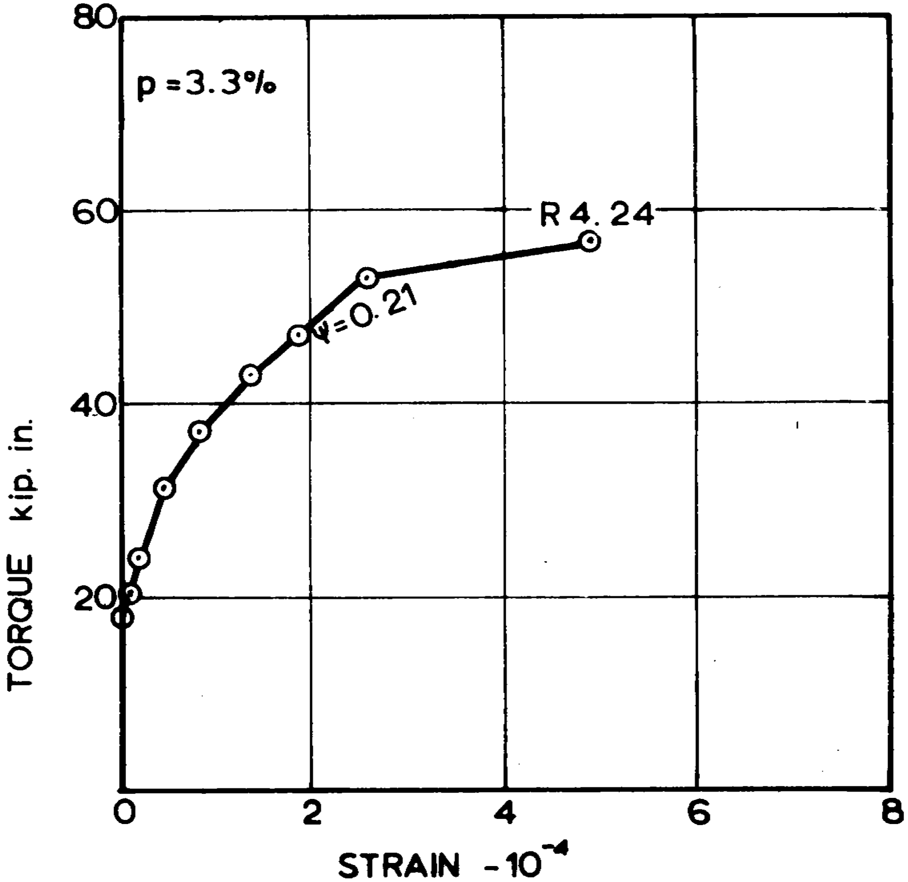


FIG.3.33 . STEEL STRAIN IN VERTICAL LEG OF HOOP.

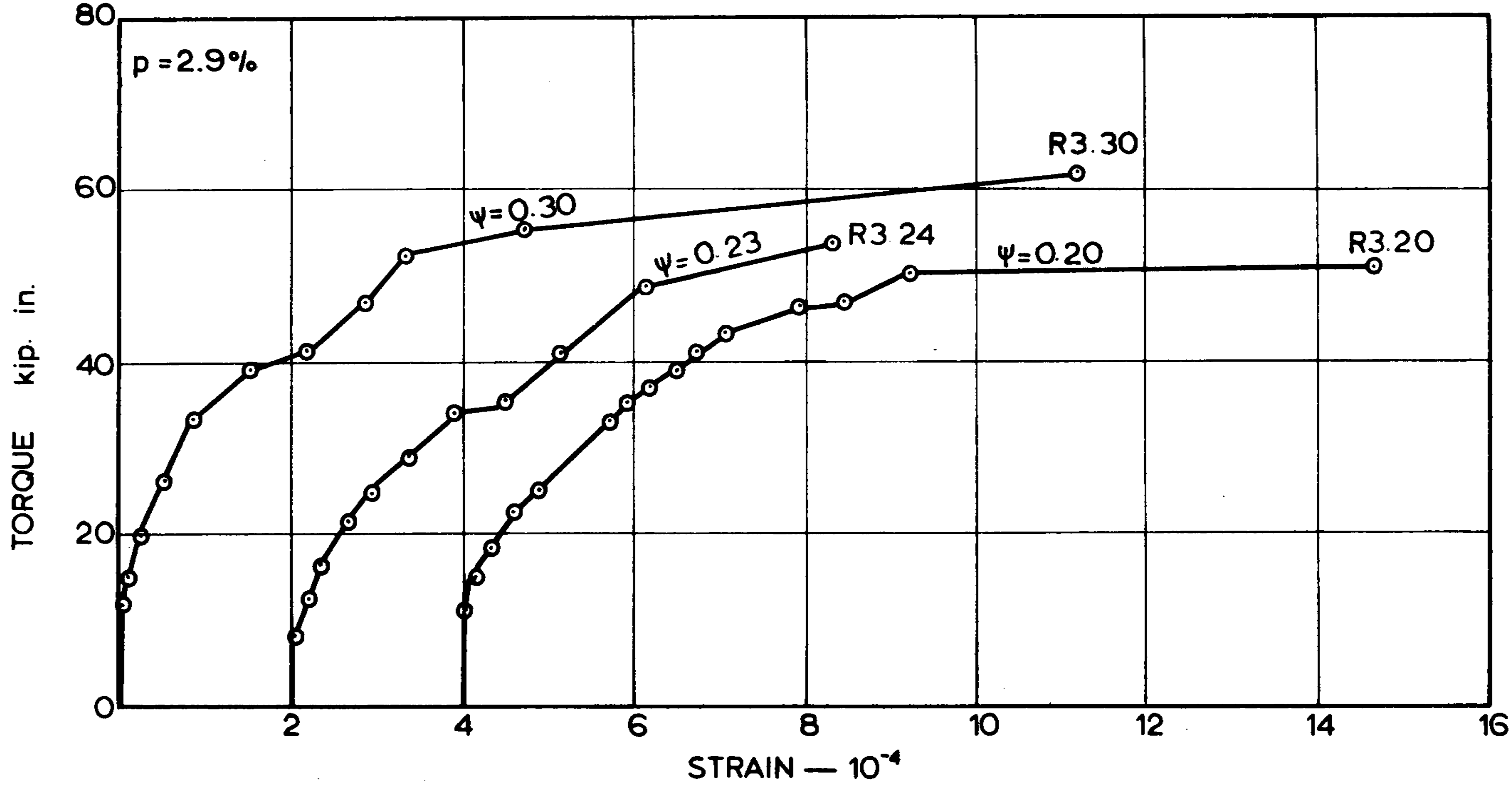


FIG. 3.33 STEEL STRAIN IN VERTICAL LEGS OF HOOPS

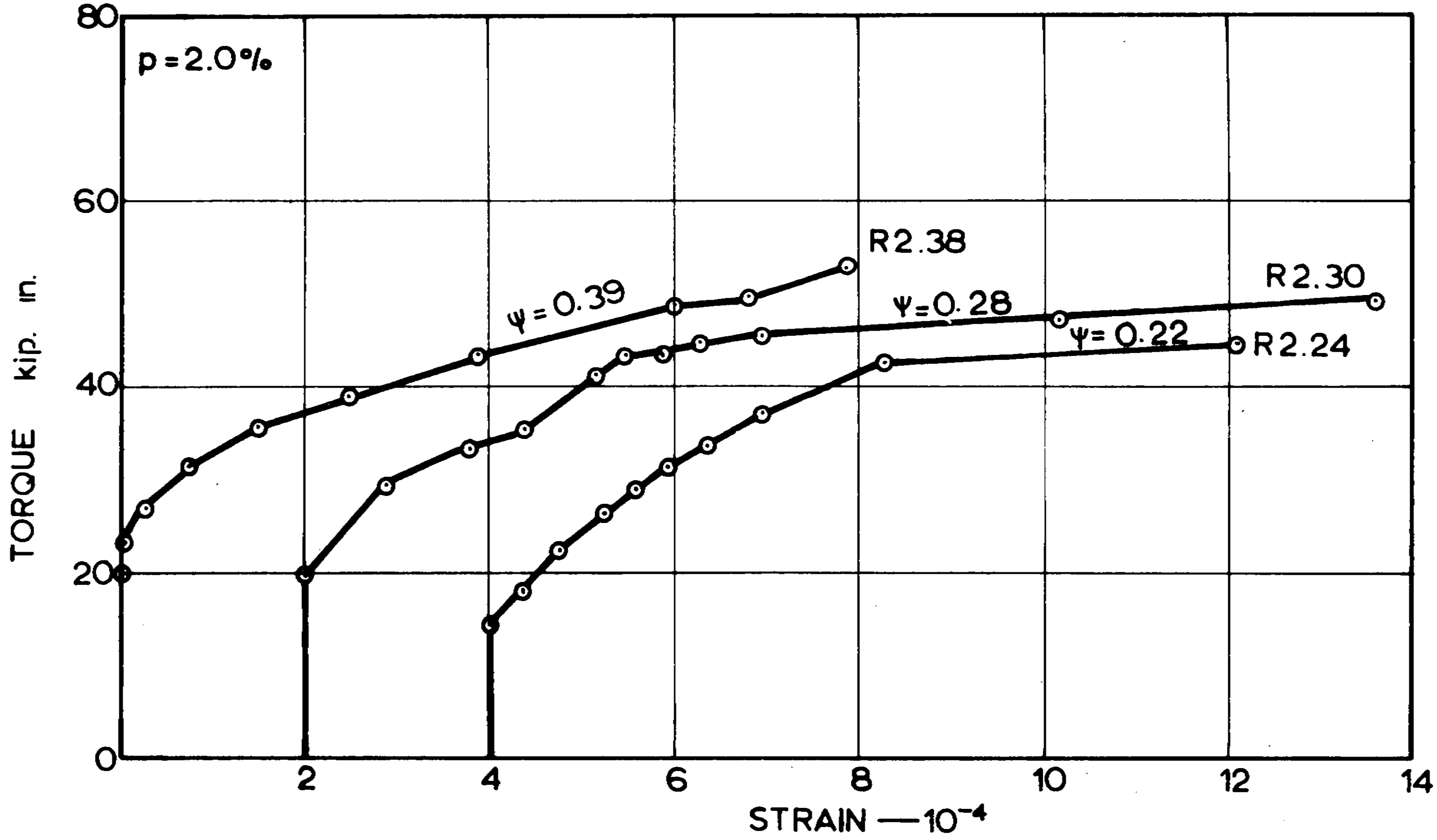


FIG. 3.33 STEEL STRAIN IN VERTICAL LEGS OF HOOPS.

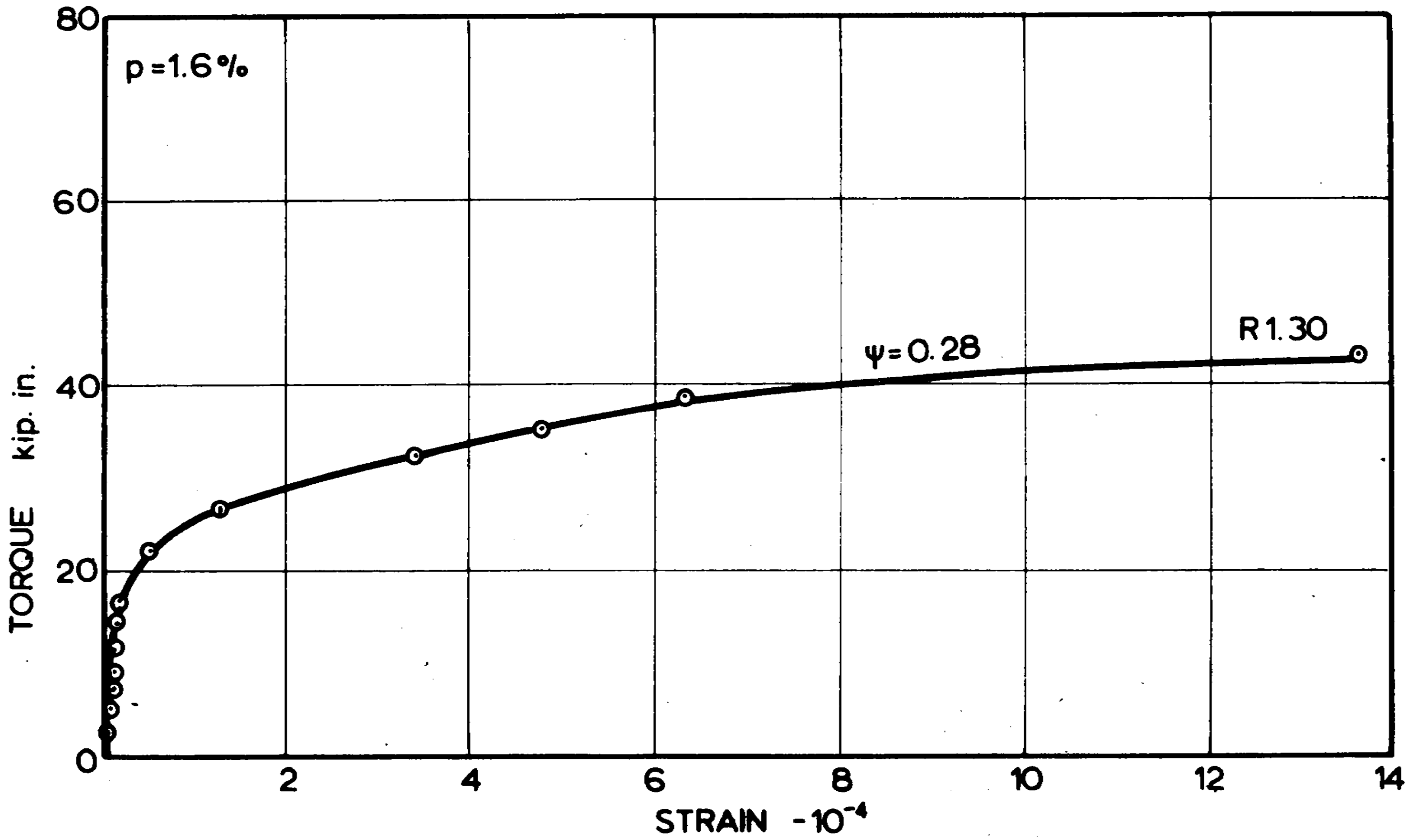


FIG 3.33 STEEL STRAIN IN VERTICAL LEGS OF HOOPS

CHAPTER 4

ULTIMATE STRENGTH OF BEAMS WITHOUT WEB REINFORCEMENT

In this chapter it will be shown that the pure torsional strength of a beam without web reinforcement may be satisfactorily computed from the plastic theory of stress distribution and an empirical tensile stress criterion of failure. The interaction of bending and torsion and shear and torsion will be discussed. Finally a simplified method, which transforms the torsional loading into an effective shear force, will be presented.

4.1 Pure Torsion

As the ultimate strength, in pure torsion, of beams containing only longitudinal steel is related to the torsional strength of plain concrete beams, it is appropriate to discuss the simpler plain specimens first.

Tests show that plain concrete specimens fail immediately on the formation of a cleavage crack normal to the principal tensile stress. The stress situation at any point in a specimen, irrespective of the stress-strain relationship or the shape of the specimen, is that of pure shear. Thus all the stress parameters commonly used in concrete failure theories bear a constant relationship to each other. For convenience, and in view of the failure behaviour, the failure criterion of plain torsion specimens may be designated as a maximum principal tensile stress criterion. It should be appreciated that the values of this limiting tensile stress for the particular stress situation of pure torsion may not correspond to values obtained for other stress situations. For a particular concrete, if scale effects can be ignored, this limiting tensile stress should be constant for all torsion specimens. Advantage may be taken of this fact in the investigation of the elastic and plastic stress distribution theories.

Tests on plain concrete sections of varying shapes but the same concrete are tabulated in Table 4.1. It can be seen from this table that the stress at failure calculated on a plastic basis is evidently

TABLE 4.1

A COMPARISON OF THE FAILURE STRESSES COMPUTED FROM THE ELASTIC AND PLASTIC THEORIES FOR BEAMS OF DIFFERENT SHAPES.

Beam	Shape	Failure Stresses (p.s.i.)		f'_c (p.s.i.)
		Elastic	Plastic	
HC	Hollow Cylinder	399	341	7,200
R	Rectangle	600	358	"
SC	Solid Cylinder	472	354	"
REP4	Rectangle	513	304	4,600
SCR	Solid Cylinder	442	332	"
RUP4	Rectangle	675	400	3,680
ELL	L-Shape	675	430	"

independent of the shape of the cross section, on the other hand the shape of the cross section has a marked effect on the indicated failure stress calculated on an elastic basis. The evidence of these few tests clearly supports the plastic theory.

To examine the use of the plastic theory and the maximum principal stress criterion for predicting first cracking, several plain concrete beams were tested in bending and torsion. For these tests the maximum principal stress was calculated from the following formula.

$$f = \frac{f_b}{2} + \frac{1}{2}\sqrt{f_b^{\,2} + 4\,\tau^2} \qquad \ldots \qquad (4.1)$$

Where f_b is the direct stress and τ is the shear stress.

If it is assumed that full plasticity provides a reasonable estimate of the stresses produced by bending and torsion, the equations below are obtained.

$$f_b \doteqdot \frac{4M}{bh^2} \qquad \ldots \qquad (4.2)$$

$$= \frac{2T}{b^2\,(h - \frac{b}{3})} \qquad \ldots \qquad (4.3)$$

The results of the bending and torsion tests are presented in Table 4.2 in three groups. Each group corresponds to beams made from the one concrete mix. If the above formulae are valid the computed principal tensile stress should be a constant for any one concrete mix. This is reasonably confirmed by examination of the results in this table.

It may be concluded that a satisfactory approach for torsion in plain concrete is a plastic stress distribution and a maximum tensile stress criterion.

Thus for rectangular sections,

$$T = \frac{1}{2}\,b^2\,(h - \frac{b}{3})\,f'_t \qquad \ldots \qquad (4.4)$$

TABLE 4.2 COMPUTED FAILURE STRESSES FOR PLAIN CONCRETE RECTANGULAR BEAMS SUBJECTED TO BENDING AND TORSION

Beam	Failure Stress in p.s.i. (corrected to f'_c = 6,720 p.s.i.)
P1	416
P2	423
P3	412
P4	378
P5	403
P6	420
mean = 409 p.s.i. mean devn. = 12 p.s.i.	

Beam	Failure Stress in p.s.i. f'_c = 4,600 p.s.i.
REP2	340
REP4	304
mean = 322 p.s.i. mean devn. = 18 p.s.i.	

Beam	Failure Stress in p.s.i. f'_c = 4,600
RUP2	366
RUP4	398
mean = 382 p.s.i. mean devn. = 16 p.s.i.	

The question of a suitable value for f'_t, the failure strength, still remains. It is the practice at present to specify the mechanical properties of concrete used for a structure solely by the compressive strength. Although it is recognised that no exact relationship exists between the tensile and compressive strengths of concrete a frequently employed approximation is,

$$f'_t = C\sqrt{f'_c} \qquad \ldots \qquad (4.5)$$

As the stress situation in torsion is the biaxial stress pattern of pure shear, it is most satisfactory to evaluate the parameter C from torsion tests. The results of all the available tests on plain torsional specimens are analysed in Table 4.3 using the plastic theory. The results indicate that this constant has a value of 5.0 ± 20%. A safe but not unduly conservative criterion can be taken as,

$$f'_t = 3.5\sqrt{f'_c} \qquad \ldots \qquad (4.6)$$

The failure torque may be obtained from the above criterion and the formula for the plastic theory, ie.,

$$T_o = 1.75\, b^2 \left(h - \frac{b}{3}\right) \sqrt{f'_c} \qquad \ldots \qquad (4.7)$$

If the results in Table 4.3 are expressed in a slightly different form the value of the ratio $T_{experimental}/T_{theoretical}$ could be obtained $(\frac{C}{3.5})$. This ratio has a mean value of 1.48 ± 20% for the results of tests on rectangular specimens and an overall mean of 1.44 ± 20% for all tests.

The reported results for torsion tests on rectangular beams with only longitudinal steel are set out in Table 4.4. These beams have been analysed by means of the formula for plain concrete given above. In this case the values of the ratio of T_{exp}/T_{theor} have a mean of 1.55 and a coefficient of variation of 18%. If this figure is compared with the result obtained for plain concrete it can be seen that the assumption that the longitudinal steel can be ignored is justified for members tested in pure torsion.

TABLE 4.3

ANALYSIS OF PURE TORSION TESTS ON PLAIN CONCRETE SPECIMENS

PART 1. RECTANGULAR SPECIMENS

Investigator	Beam	Torque kip.in.	f'_c p.s.i.	Plastic Failure Stress	C	$\frac{T_{exp.}}{T_{theor.}}$
This Investigation	R1	13.40	7200	358	4.23	1.21
	REP4	50.50	4600	305	4.50	1.29
	RUP4	66.30	3680	400	6.60	1.89
	W1	73.20	4630	389	5.72	1.63
	W2	74.50	4320	450	6.85	1.96
Bach and Graf		150.50	2830	274	5.15	1.47
		128.90	2830	274	5.16	1.47
Young, Sagar and Hughes	A1	13.90	1700	333	8.09	2.31
	2	20.20	1700	287	6.97	1.99
	3	37.40	1700	448	10.88	3.11
Turner and Davies	S 1	11.00	2400	263	5.39	1.54
	2	11.75	2400	281	5.76	1.64
	R1	17.00	2400	318	6.51	1.86
	2	12.00	2400	224	4.59	1.31
Andersen	3A	55.00	4100	322	5.03	1.44
	B	73.00	4100	311	4.86	1.39
	C	120.00	4100	401	6.27	1.79
	4A	67.00	6900	392	4.73	1.35
	B	88.00	6900	374	4.51	1.29
	C	117.00	6900	391	4.72	1.35
Marshall and Tembe	O1	7.59	1430	182	4.82	1.38
	2	7.28	1430	174	4.62	1.32
	3	6.91	1430	165	4.39	1.25
	4	7.69	1430	184	4.88	1.39
	5	8.70	2560	208	4.13	1.18
	5	10.56	2560	253	5.01	1.43
	5	9.24	3000	221	4.05	1.16
	8	8.70	3000	208	3.81	1.09
	8	9.22	3120	221	3.96	1.13
	10	9.22	3120	221	3.96	1.13

Table 4.3 Contd.

Investigator	Beam	Torque kip. in.	f'_c p.s.i.	Plastic Failure Stress	C	$\frac{T_{exp.}}{T_{theor}}$
Marshall and Tembe (contd.)	A1	10.25	2560	274	5.43	1.55
	2	9.74	2560	260	5.16	1.47
	3	10.25	2560	274	5.43	1.55
	4	10.30	2560	275	5.45	1.56
	7	6.15	2560	288	5.70	1.63
	8	7.18	2560	336	6.65	1.90
	9	9.52	2560	446	8.82	2.52
	10	9.22	2560	432	8.54	2.44
	11	6.86	2560	321	6.36	1.82
	12	7.06	2560	330	6.54	1.87
Nylander	I 1	46.80	3580	288	4.81	1.38
	2	52.61	3580	323	5.41	1.55
	II 5	41.60	3580	256	4.28	1.22
	6	36.40	3580	224	3.74	1.07
Cowan	X	38.50	3390	267	4.59	1.31
	T1	58.10	6200	461	5.86	1.67
Humphreys	PO A	20.20	7000	484	5.79	1.66
	B	19.40	7000	465	5.56	1.59
	C	20.10	7000	482	5.77	1.65
	D	19.70	7000	472	5.65	1.61
	E	19.90	7000	477	5.71	1.63
	PROA	48.10	7000	461	5.52	1.58
	B	41.80	7000	401	4.80	1.37
	C	42.40	7000	407	4.87	1.39
	PRHA	42.10	7000	404	4.83	1.38
	B	44.50	7000	427	5.11	1.46
	C	44.00	7000	422	5.05	1.44
	PS.OA	69.00	7000	413	4.95	1.41
	B	68.40	7000	410	4.91	1.40
	C	69.00	7000	413	4.95	1.41

Table 4.3 Contd.

Investigator	Beam	Torque kip. in.	f'_c p.s.i.	Plastic Failure Stress	C	$\frac{T_{exp.}}{T_{theor}}$
Humphreys	PTOA	16.60	7000	461	5.51	1.57
	B	15.90	7000	441	5.28	1.51
	C	14.70	7000	408	4.88	1.39
	PUOA	22.20	7000	448	5.36	1.53
	B	21.70	7000	438	5.24	1.50
	C	22.60	7000	456	5.46	1.56
	RP1	28.55	6000	334	4.32	1.23
	2	26.36	6200	308	3.92	1.12
	3	24.24	6350	284	3.56	1.02
	4	27.44	6350	321	4.04	1.15
	5	28.04	6400	328	4.11	1.17
	6	32.04	6400	375	4.69	1.34
	7	28.24	7180	330	3.91	1.12
	8	29.76	6500	348	4.33	1.24
	9	31.24	6950	366	4.39	1.25
				Mean	5.20	1.48
				Standard Deviation	20 °/o	20 °/o
				No. of Tests	75	75

PART 2. CIRCULAR SPECIMENS

Investigator	Beam	Torque kip. in.	f'_c p.s.i.	Plastic Failure Stress	C	$\frac{T_{exp}}{T_{theor}}$	Comment
This Investigation	HC	13.6	7200	341	4.03	1.15	Hollow
	SC	20.0	7200	353	4.17	1.18	Solid
	SCL	20.8	6660	367	4.51	1.29	"
	SCR	18.8	4600	332	4.90	1.40	"
Graf and Morsch	1A	130.0	1780	167	3.97	1.13	Hollow
	1B	140.3	"	180	4.28	1.22	"
	1C	108.2	"	139	3.30	0.95	"
	2A	216.5	"	211	5.02	1.43	Solid
	2B	216.5	"	211	5.02	1.43	"
	2C	173.2	"	169	4.01	1.14	"
Miyamoto	GRP1	11.3	1821	277	6.49	1.85	Solid
Andersen	R1	25.2	2000	188	4.20	1.20	Solid
	R2	30.8	2100	229	5.01	1.43	"
	R3	35.2	2980	262	4.81	1.37	"
	R4	29.8	3200	222	3.93	1.12	"
	R5	32.2	3590	240	4.01	1.14	"
	R6	42.2	5200	314	4.37	1.25	"
				Mean	4.52	1.29	
				Standard Deviation	15 °/o	15 °/o	
				No. of Tests	17	17	

Analysis of All Results	$\frac{T_{exp}}{T_{theor}}$	C
Mean	1.44	5.0
Standard Deviation	20 °/o	20 °/o
No. of Tests	92	92

TABLE 4.4

A COMPARISON OF THEORY AND EXPERIMENT FOR BEAMS CONTAINING ONLY LONGITUDINAL REINFORCEMENT LOADED IN PURE TORSION

				T_{exp}/T_{theor}	
Investigator	Beam	Torque kip. in.	f'_c p.s.i.	Plastic Eq. (4.7)	Eff. Shear see Sect. 4.4
Bach and Graf		156.0	2820	1.52	1.58
		173.4	"	1.70	1.76
		160.2	"	1.58	1.62
		160.2	"	1.58	1.62
		180.8	"	1.76	1.84
		173.4	"	1.70	1.76
		130.0	"	1.49	1.93
		136.8	"	1.57	2.02
		136.8	"	1.57	2.02
		141.0	"	1.61	2.09
		130.0	"	1.49	1.93
		141.0	"	1.61	2.09
Young Sagar and Hughes	B1	14.0	1700	2.33	2.40
	B2	22.7	"	2.15	2.61
	B3	36.7	"	2.40	3.14
Turner and Davies	S3	12.5	2400	1.76	1.81
	S7	12.0	"	1.32	1.70
Andersen	B, 1	69.8	2100	1.30	1.36
	2	77.1	2250	1.39	1.45
	3	80.1	2250	1.44	1.50
	4	84.5	3600	1.21	1.25
	5	88.5	3600	1.27	1.31
	6	97.6	3680	1.37	1.43
	7	105.1	5000	1.27	1.33
	8	109.2	5000	1.32	1.38
	9	119.9	5200	1.42	1.48

Table 4.4 Contd.

				T_{exp}/T_{theor}	
Investigator	Beam	Torque kip. in.	f'_c p.s.i.	Plastic Eq. (4.7)	Eff. Shear see Sect. 4.4
Andersen	1A	50.0	3900	1.34	1.39
	1B	73.0	3900	1.43	1.62
	1C	90.0	3900	1.23	1.28
	2A	62.0	7000	1.24	1.29
	2B	98.0	7000	1.43	1.63
	2C	122.0	7000	1.25	1.30
Marshall and Tembe	B1	11.8	2560	1.79	2.15
	B2	11.3	2560	1.71	2.06
	B3	11.8	2560	1.79	2.15
	C1	11.3	2560	1.71	2.06
	C2	11.9	2560	1.80	2.16
	C3	10.8	2560	1.65	1.99
Nylander	III 1A	13.0	2859	1.50	1.99
	III 1B	13.0	2859	1.50	1.99
	IV 5A	15.6	3079	1.73	2.30
	IV 5B	14.7	3079	1.64	2.17
Cowan	A	36.0	3380	1.22	1.52
Ernst	3TRO	37.6	3923	0.95	1.23
	4TRO	34.4	"	0.88	1.13
	5TRO	33.8	"	0.86	1.11

Table 4.4 Contd.

				T_{exp}/T_{theor}	
Investigator	Beam	Torque kip. in.	f'_c p.s.i.	Plastic Eq. (4.7)	Eff. Shear see Sect. 4.4
Humphreys	P1 A	19.7	7000	1.62	1.68
	B	21.5	"	1.76	1.82
	C	22.2	"	1.82	1.89
	D	21.1	"	1.73	1.80
	E	19.3	"	1.57	1.65
	PRI A	44.8	"	1.47	1.90
	B	46.1	"	1.52	1.96
	C	45.8	"	1.52	1.95
	PSI A	74.0	"	1.52	2.10
	B	73.5	"	1.50	2.08
	C	72.0	"	1.47	2.04
	PTI A	16.5	"	1.56	2.16
	B	15.5	"	1.47	2.03
	C	14.9	"	1.42	1.95
	PUI A	24.7	"	1.71	2.42
	B	21.2	"	1.47	2.08
	C	22.2	"	1.53	2.18
Gesund and Boston	1	36.0	4379	0.91	0.95
	2	39.0	4379	0.99	1.03
Ramakrisnan and Vijarangan		28.8	3119	1.86	2.30
		28.8	3099	1.87	2.30
		26.1	2639	1.83	2.28
		23.2	2179	1.79	2.23
		20.1	1969	1.63	2.03
		21.7	2000	1.75	2.15
			Mean	1.55	1.84
			Standard Deviation	18°/o	22°/o
			No. of Tests	71	71

4.2 Combined Bending and Torsion of Beams Containing only Longitudinal Steel

Beams containing only longitudinal steel and loaded in combined flexure and torsion have been observed to fail in several modes. (Chapter 3). Consideration of these observed failure patterns suggests that the ultimate strength of these beams would depend on factors such as, the ratio of bending moment to twisting moment, the amount and distribution of longitudinal steel, the dowel forces between the steel and the concrete, the strength of the concrete under combined stresses and the magnitude of the aggregate interlock forces. Attempts have been made by Gesund and Boston, and Nylander to analyse possible failure mechanisms. However because of the assumptions necessary in their analyses, neither method gives a satisfactory estimate of the failure load.

If web reinforcement is not used in the design of a beam to resist torsion combined with other loadings, invariably the capacity of the section under torsion and shear will govern the size of the member. It was considered, therefore, that a simple empirical approach would be adequate for the case of bending and torsion.

To investigate the interaction behaviour of this type of beam Figure 4.1 was prepared. In this figure, the ratio of the observed failure moment to the ultimate pure flexural strength is plotted against the ratio of the observed failure torque to the pure torsional strength for available test results. A considerable range of section properties is encompassed in these tests, the ratio of height to width varying from 1.5 to 2.1, the percentage of steel from 1.1% to 2.4% and the concrete cylinder strength from 2000 p.s.i. to 7200 p.s.i. For these beams the pure torsional strength was determined from tests on companion specimens.

It can be observed from this figure that the presence of bending moment has little effect on the magnitude of the torsion strength. From this it follows that the torsional strength of a beam loaded in flexure and torsion may be found by calculating the pure torsional strength

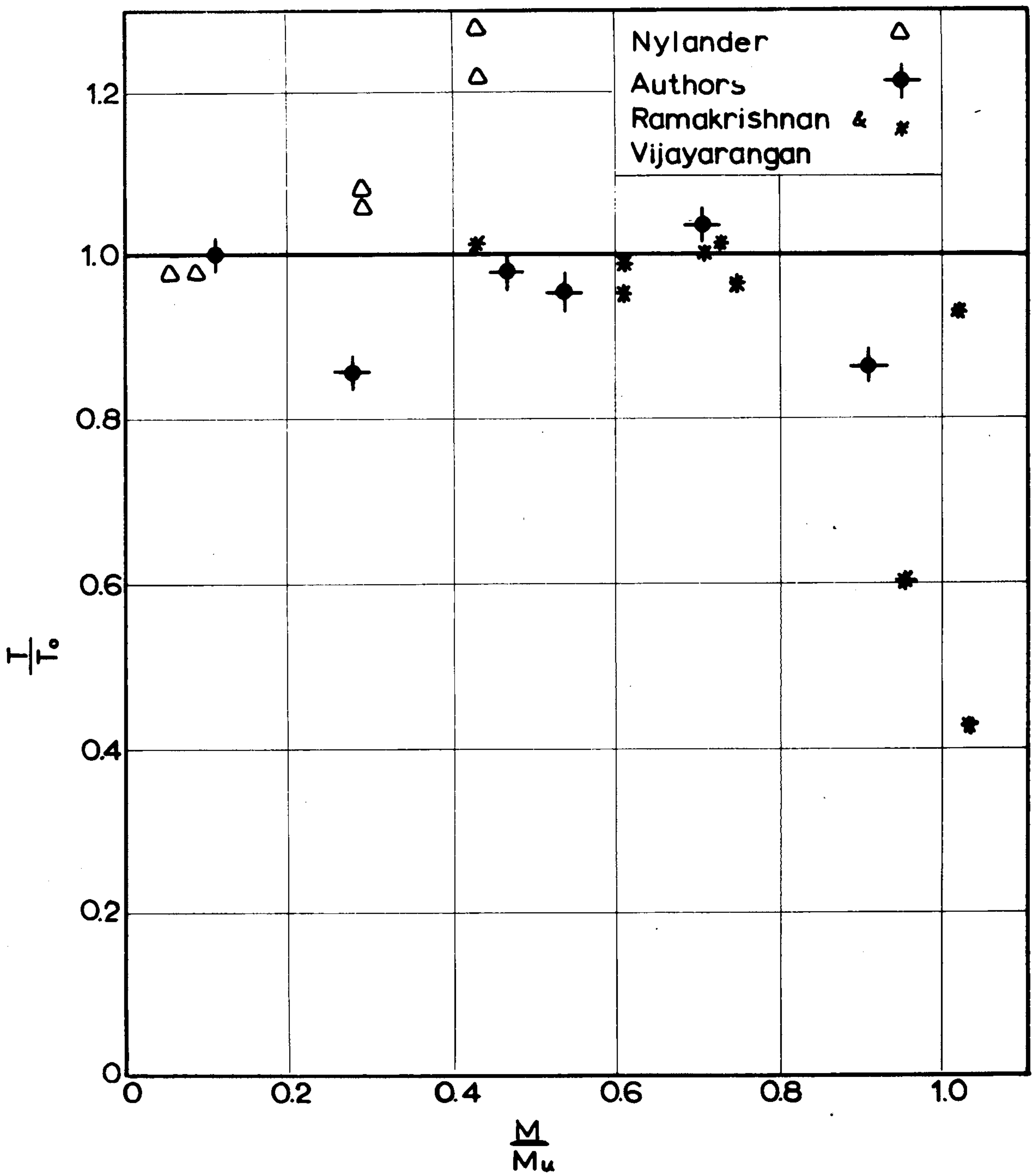


FIG.4.1 FLEXURE TORSION INTERACTION FOR BEAMS CONTAINING ONLY LONGITUDINAL STEEL.

$$\text{i.e.}\quad T_o = 1.75\, b^2 \left(h - \frac{b}{3} \right) \sqrt{f'_c} \qquad \ldots \qquad (4.7)$$

The accuracy of this assumption is demonstrated in Table 4.5. The mean value of T_{exp}/T_{theor} for the 43 available results is 1.49 with a standard deviation of 18%.

It may be recalled that the mean value of T_{exp}/T_{theor} for rectangular beams with only longitudinal reinforcement loaded in pure torsion was 1.55 ± 18%. Comparison of these values also supports the assumption that the effect of bending on the torsional strength can be ignored.

4.3 Shear and Torsion in Beams with only Longitudinal Reinforcement

As a result of Nylander's work (Chapter 2) it has generally been assumed that a maximum principal stress approach satisfactorily accounts for the strength of beams loaded in shear and torsion. Thus a beam is assumed to fail when the sum of torsional shearing stress and the direct shear stress equals the tensile strength of the concrete. A typical example of this method, in working stress form, is given in the SAA code "Concrete in Buildings" (CA2). The S and LS series of tests described in Chapter 3 may be used to investigate this method.

The results of these tests are presented in Figure 4.2, in which the failure stress obtained from the equations given in SAA code due to both torsion and shear are plotted against V/V_c. It should be noted that all beams of this series were cast from the one batch of concrete, so the tensile strength should be sensibly constant. In fact the indicated failure stress is seen to vary markedly with the proportion of shear in the combined loading. For these beams the factor of safety varied from 2.37 for pure shear to 9.03 for pure torsion. It must be concluded that the maximum principal stress approach on which the SAA code rules are based is unsatisfactory for combined shear and torsion.

A more satisfactory procedure can be developed if an empirical interaction relationship between shear and torsion is employed. The observed shear-torsion interaction behaviour is shown in Figure 4.3, where the ratio

TABLE 4.5

A COMPARISON OF THE THEORY WITH EXPERIMENTAL RESULTS FOR BEAMS CONTAINING ONLY LONGITUDINAL STEEL LOADED IN BENDING AND TORSION

				$T_{exp}/T_{theor.}$	
Investigator	Beam	Torque	Moment	Plastic Eq.(4.7)	Eff. Shear see Sect. (4.4)
This investigation	L1	57.0	32.3	1.62	2.12
	L2	60.1	205.7	1.68	2.19
	L3	61.1	5.5	1.63	2.13
	L7	47.8	80.5	1.38	1.80
	L8	49.6	263.5	1.40	1.83
	LS2	52.5	5.2	1.87	2.54
	S3	64.2	6.3	1.74	2.19
	LB1	54.1	141.6	1.47	1.90
	LB2	55.4	6.3	1.50	1.91
	LB3	60.7	161.5	1.65	2.14
Ramakrishnan and Vijarangan	B4	17.1	99.0	1.36	1.52
	B5	24.8	45.4	1.85	2.56
	B6	10.7	108.0	1.39	1.39
	C3	21.7	111.0	1.69	2.34
	C4	20.1	90.7	1.63	2.26
	C5	23.2	105.0	1.79	2.47
	$B6_B$*	23.2	108.0	1.70	2.35
	$C1_B$*	23.2	108.0	1.79	2.48
	$C6_B$*	21.7	90.7	1.75	2.42

Table 4.5 Cont.

Investigator	Beam	Torque	Moment	$T_{exp}/T_{theor.}$ Plastic Eq. (4.7)	$T_{exp}/T_{theor.}$ Eff. Shear see Sect. (4.4)
Nylander	III 2A	13.0	9.2	1.46	2.03
	2B	13.0	9.2	1.46	2.03
	3A	14.3	48.4	1.65	2.28
	3B	13.9	48.4	1.60	2.22
	4A	16.5	72.6	1.94	2.69
	4B	15.6	72.6	1.84	2.54
	VII 1	39.0	52.1	1.19	1.30
	2	31.2	52.1	0.95	1.04
	3	39.1	58.0	1.19	1.30
	4	35.2	58.0	1.07	1.17
	5	54.7	75.5	1.70	1.86
	6	50.7	75.5	1.58	1.73
	7	50.7	10.0	1.51	1.65
	8	54.7	10.0	1.63	1.78
	9	31.3	58.0	0.94	1.03
	10	19.5	58.0	0.96	0.96
Gesund and Boston	3	58.0	58.0	1.47	1.70
	4	64.0	64.0	1.62	1.89
	5	43.0	86.0	1.49	1.74
	6	36.0	108.0	1.25	1.46
	7	59.0	177.0	1.32	1.54
	8	49.0	195.0	1.10	1.28
	9	42.0	83.0	0.97	1.47
	10	39.0	156.0	1.17	1.77
			Mean	1.49	1.86
			Standard Deviation	18 °/o	24 °/o
			No. of Tests	43	43

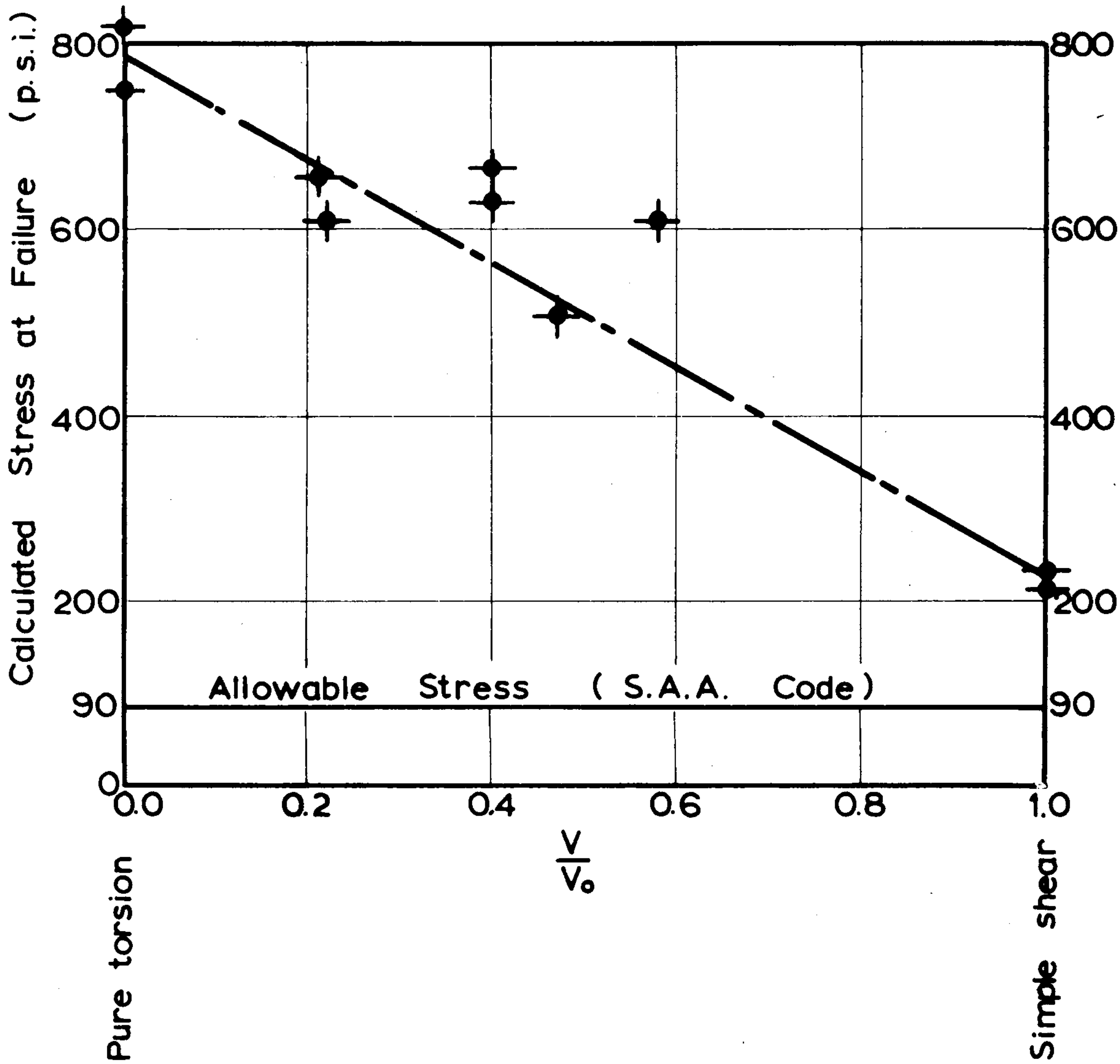


FIG. 4.2 THE EFFECT OF THE RATIO OF LOADING ON THE CALCULATED STRESS AT FAILURE.

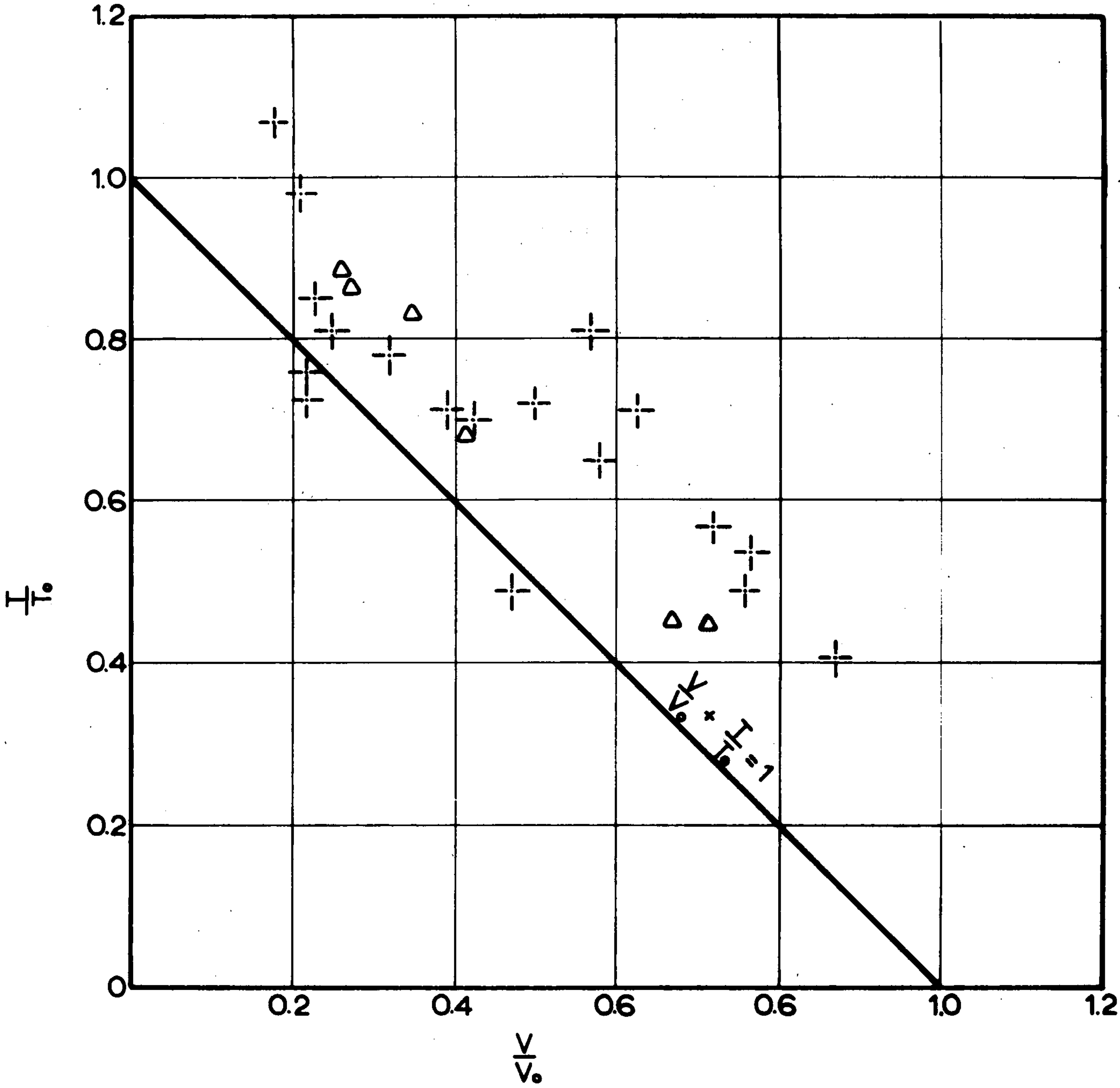


FIG. 4.3. SHEAR TORSION INTERACTION FOR BEAMS CONTAINING ONLY LONGITUDINAL STEEL.

of the failure torque to the pure torsional strength, is plotted against the ratio of the failure shear to the pure shear strength. For the results plotted both pure torsional and shear strengths were determined from tests on companion specimens.

It is evident that the assumption of linear interaction between shear and torsion closely follows the trend of the experimental data, hence

$$\frac{T}{T_o} + \frac{V}{V_o} = 1 \qquad \ldots\ (4.8)$$

4.4 Effective Shear Method

Although the equations developed in the previous sections are relatively simple, there are certain advantages to be gained from recasting the equations into a simpler form. The main purpose of this is to make the equations for beams without web reinforcement consistent with the design procedure that will be derived for beams with web reinforcement.

The A.C.I. code (316-63) gives the following simplified formula for the shear capacity of an unreinforced section,

$$V_o = 2bd\sqrt{f'_c} \qquad \ldots\ (4.9)$$

The expression for the torsional strength as given in section 4.1 is,

$$T_o = 1.75\, b^2 (h - \frac{b}{3}) \sqrt{f'_c}$$

This equation may be rearranged to give,

$$T_o = \frac{\dfrac{(2bd\sqrt{f'_c})b}{2d}}{1.75(h - \frac{b}{3})}$$

A reasonable approximation to this formula will be given by,

$$T_o = \frac{V_o b}{1.6} \qquad \ldots\ (4.10)$$

The above relationship will give values for T_o comparable with equation (4.7) when $\frac{h}{b} = 1$ and more conservative values when $\frac{h}{b} > 1$.

To use this equation to analyse pure torsion results it is suggested that d be taken as 0.9h. The accuracy of this method for the case of pure torsion is indicated in Table 4.4. The mean of the values of T_{exp}/T_{theor} for the 75 available tests is 1.84 ± 22%. For bending and torsion tests the results shown in Table 4.5 yield a mean of 1.86 ± 24%. The modified method is somewhat more conservative than the plastic equation (4.7), although it is sufficiently accurate for design use.

For the case of shear and torsion, the linear interaction proposed in section 4.3 is expressed in the form,

$$\frac{T}{T_o} + \frac{V}{V_o} = 1$$

When the value for T_o from equation (4.10) is substituted in the above expression we obtain,

$$V + \frac{1.6T}{b} = V_{eff} = V_o \qquad \ldots \qquad (4.11)$$

Use of this equation in analysing the results contained in Table 4.6 gives a mean of 2.22 ± 16% for the values of T_{exp}/T_{theor}.

Equation (4.11) in effect means that the torque on a member loaded in combined torsion, bending and shear, can be replaced by an equivalent shear of 1.6 T/b. Design is then carried out for the applied bending moment and an effective shear by the usual methods.

TABLE 4.6

A COMPARISON OF THE THEORY WITH EXPERIMENTAL RESULTS FOR BEAMS CONTAINING ONLY LONGITUDINAL REINFORCEMENT LOADED IN SHEAR AND TENSION

					T_{exp}/T_{theor}	
Investigator	Beam	Torque kip.in.	Shear kips.	$\frac{M}{Vd}$	Plastic eq. (4.8)	Eff. Shear see Sect. 4.4
This investigation	L5	43.4	4.0	4.52	1.75	2.01
	L6	52.5	0.8	5.01	1.54	1.86
	SL1	46.2	2.2	4.45	2.21	2.53
	LS3	39.6	1.9	3.74	1.71	2.00
	LS4	25.7	4.2	3.71	1.59	1.77
	LS6	37.6	3.6	3.76	1.91	2.19
	S1	41.8	6.4	3.07	1.93	2.08
	S4	47.0	4.5	2.97	1.67	1.80
	S5	47.7	2.4	3.10	1.60	1.77
	Q1	13.9	5.6	5.37	1.68	1.87
	Q2	24.1	4.2	5.25	1.89	2.21
	Q4	27.8	1.6	5.21	1.57	1.94
	Q4A	27.3	4.3	4.07	2.04	2.37
	Q5	16.5	5.7	4.07	1.80	2.00
	Q6	19.4	6.0	3.06	1.95	2.15
	Q6A	24.4	4.3	3.05	1.87	2.12
	Q7	28.8	1.7	4.01	1.63	1.98
	Q7A	27.3	4.3	4.07	2.04	2.37
	Q8A	33.0	1.8	2.98	1.83	2.16
	Q9	18.3	5.0	5.42	1.77	2.02
	Q10	36.4	2.2	2.03	2.04	2.28
	Q11A	26.5	4.0	2.03	1.88	2.06
	Q11B	27.2	10.0	2.05	2.91	3.09
Nylander	IV2A	6.9	4.3	4.4	2.34	2.66
	2B	6.8	4.1	4.4	2.25	2.56
	3A	10.3	2.5	4.4	2.07	2.55
	3B	12.7	2.1	4.4	2.15	2.74
	4A	13.4	1.6	4.4	2.01	2.60
	4B	13.6	1.6	4.4	2.00	2.61
		Mean			1.93	2.22
		Standard Deviation			15 °/o	16 °/o
		No. of Tests			29	29

CHAPTER 5

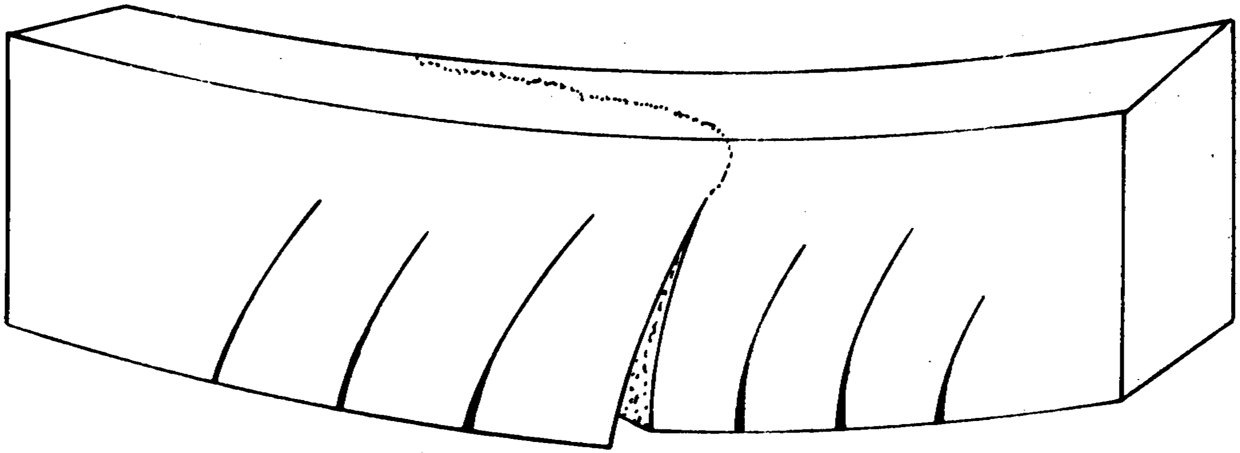
ULTIMATE STRENGTH THEORY FOR BEAMS CONTAINING BOTH LONGITUDINAL AND TRANSVERSE STEEL

There is universal agreement among workers in the field that combinations of longitudinal and transverse steel increase the torsional capacity of beams. Advantage may be taken of this fact to reduce the overall dimensions of beams subjected to torsion in combination with other actions.

In recent years several investigators have proposed theories to calculate the ultimate strength of beams of this type. It is generally agreed that failure of the beams takes place in the manner suggested in Figure 5.1. The beam fails when tension cracks on three of the sides open allowing the segments of the beam to rotate about an axis located near the fourth side. In analyzing this mechanism various investigators have made differing assumptions. For example the direction of the axis about which the beam rotates has been taken as parallel to the longitudinal axis of the beam, Gesund, joining the ends of the tension spiral, Lessig and Yudin, or at 45^{o} to the longitudinal axis of the beam, Evans. Each investigator has made further assumptions as to the shape of the tension spiral and the depth of the compression zone. From observations of the failure of beams (Chapter 3) the author has adopted a mechanism similar in shape to that of Lessig's and Yudin's.

The various forces acting in the mechanism at failure are shown in Figure 5.2. Shearing stresses in the plane of the compression zone produce a torque T_c and a shear V_c as shown in the figure. If all six equations of equilibrium are considered these secondary forces could be evaluated.

In the author's analysis of this mechanism the only equilibrium equation employed in obtaining the design formulae is that relating



FAILURE SURFACE UNDER COMBINED BENDING & TORSION

FIG 5.1

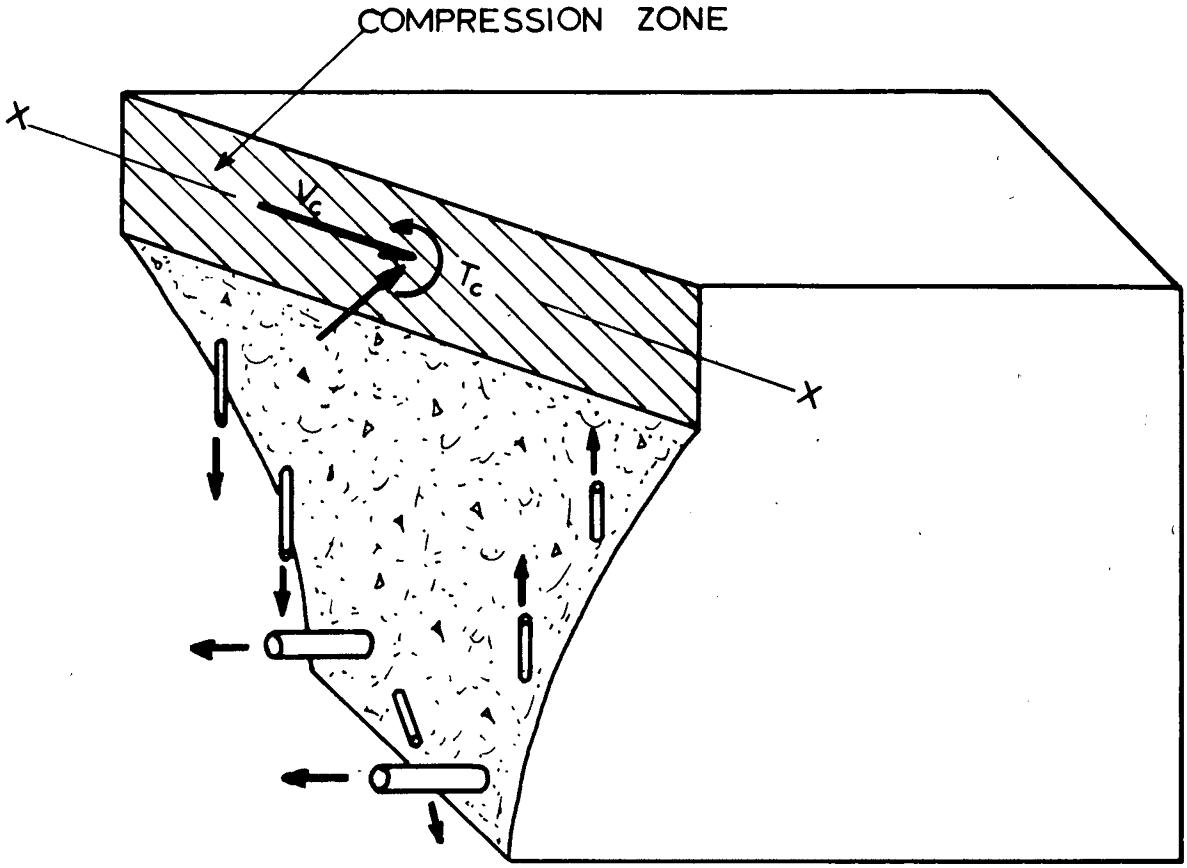


FIG. 5.2 FORCES IN FAILURE MECHANISM

moments of external and internal forces about an axis parallel to the neutral axis. This axis is designated XX' in Figure 5.2. The secondary forces T_c and V_c have no moment about this axis.

To compute the depth of compression allowing for the effect of shearing stresses on the compressive strength of concrete would require the values of T_c and V_c and the pattern of stresses produced by them. However it is considered that it is not necessary that the depth of the compression zone should be known exactly, as even considerable variation in this depth will have little effect upon the value of the moment. Equally the actual distribution of the stresses in the compression zone is not important. When the compression zone occurs near the top face of the beam it has been assumed that the depth to the compression resultant may be taken as the same as it would be in pure flexure.

It has been assumed that the angle of the tension crack on the surface opposite the compression zone could be defined by a spiral joining the ends of the compression zone at a constant angle to the longitudinal axis of the beam.

Three principal modes of failure have been observed for beams in which the steel yields. Failure with the neutral axis occurring near the top face is referred to as a mode 1 failure, near the side face as a mode 2 failure, whilst a mode 3 failure indicates that the axis forms near the bottom surface (see Figure 5.3). A full description of the failure behavious of such beams is given in Chapter 3.

Dowel forces are ignored in this analysis, and the contribution of the tensile stresses in the concrete is also omitted. These approximations lead to satisfactory results except for beams in which the amount of transverse steel is very small. For such beams the theory will frequently lead to a low estimate of the torsional capacity. This deficiency in the theory will be referred to again in a later section.

Under certain ratios of load and certain arrangements of reinforcement failure may occur before the longitudinal and the transverse steel have yielded. For such beams an attempt has been made

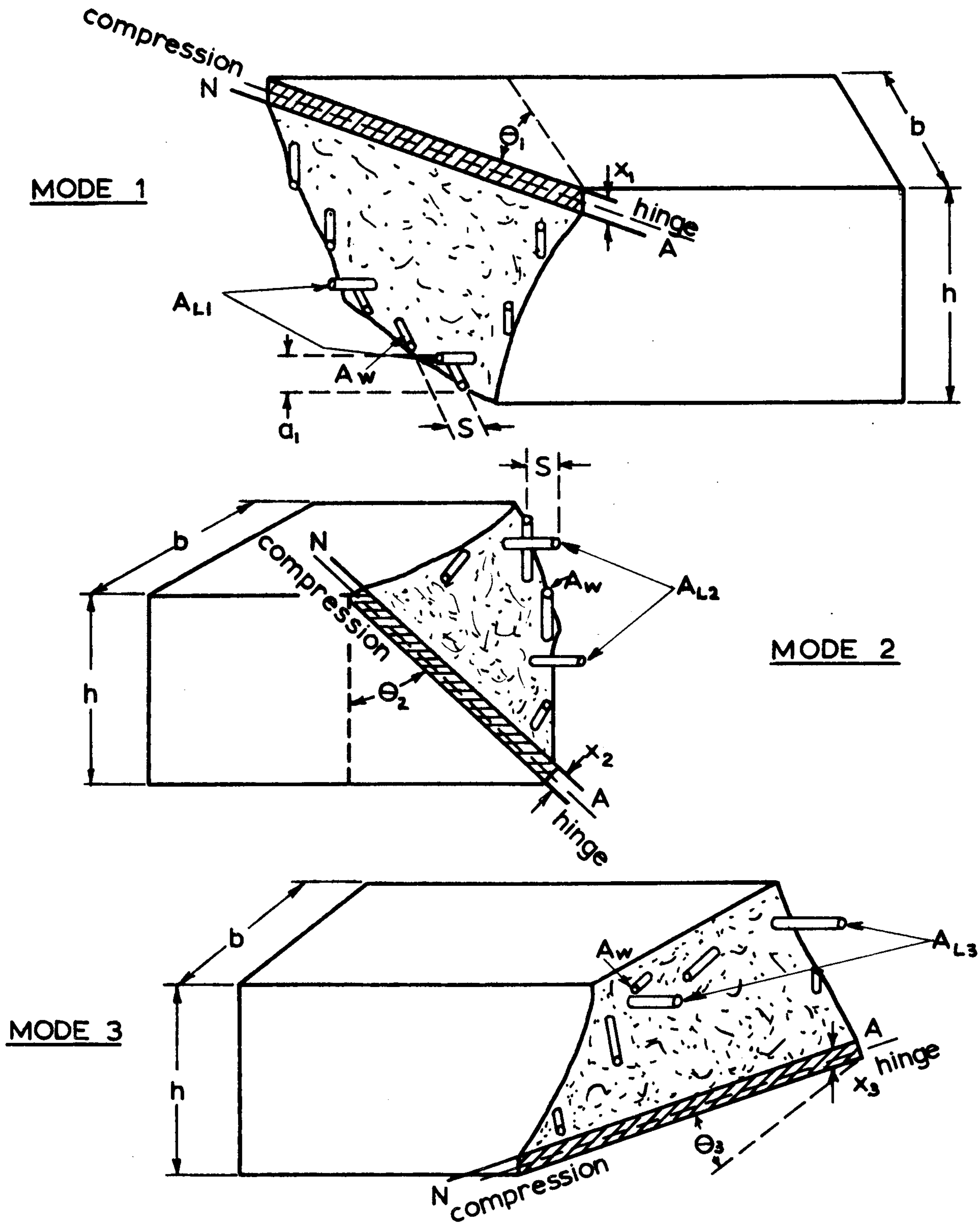


FIG. ~~3.13~~ 53 IDEALIZED FAILURE MODES either to estimate the failure load by some other means or to limit the applicability of the theory to exclude these cases. A particular example of this type of failure, is a premature failure of the beam in shear prior to the full development of a mode 2 mechanism.

It is convenient to divide the presentation of the ultimate strength theory into two sections; bending and torsion, and shear and torsion. In the section on bending and torsion, mode 1 and mode 3 failures will be discussed, while in the section on shear and torsion, the mode 2 mechanism and premature shear failures will be dealt with.

5.1 BENDING AND TORSION

Under normal circumstances the strength of a web-reinforced beam in bending and torsion will depend upon its resistance to a mode 1 failure. This type of failure mechanism is perhaps more easily visualised as a distortion of the usual pure bending failure surface. The idealized form of this failure surface is shown in Figure 5.3. On the basis of the assumptions discussed above, it is possible to analyse this mechanism to obtain an expression for the failure load.

5.1(a) Mode 1

The total moment of the internal forces about the "compression hinge" is equated to the moment of the external forces. For a mode 1 failure, the only internal forces which have a significant moment about this axis are:

(a) the forces in the bottom longitudinal steel, and
(b) the forces in the bottom branches of the transverse steel.

The forces in the vertical legs of the transverse steel have been ignored in the failure mode because their contribution to moment about the failure axis is small. This simplifies the equations and leads to conservative errors of less than 10% for most beams.

For simplicity it is also assumed that the bottom branches of the stirrups are at the same level as the main longitudinal steel.

If the angle between the "compression hinge" and the normal cross-section is θ_1, (see Figure 5.3), the total moment of the external forces about the compression hinge is

$$M_{ext} = M \cos \theta_1 + T \sin \theta_1$$
$$= T \left(\frac{\cos \theta_1}{\psi} + \sin \theta_1\right) \qquad \ldots \quad (5.1)$$

where $\psi = T/M$ $\qquad \ldots \quad (5.2)$

The area of the bottom branches of stirrups, intercepted by the failure surface is

$$\frac{A_w}{s} \frac{(b - 2a_3)\, b \tan \theta_1}{b + 2h}$$

The total moment of the internal forces about the compression hinge is then

$$M_{int} = A_{L1}f_{L1}(h - a_1 - x_1/2) \cos \theta_1 + \frac{A_w f_w}{s} \cdot \frac{b \tan \theta_1 (b-2a_3)(h-a_1-x_1/2)\sin \theta_1}{b + 2h}$$

If the relationship between transverse steel and longitudinal steel is expressed by a parameter r, where

$$r = \frac{A_w f_w}{s} \cdot \frac{(b - 2a_3)}{A_{L1}f_{L1}} \qquad \ldots \quad (5.3)$$

then equilibrium can be expressed in the form

$$T\left(\frac{\cos \theta_1}{\psi} + \sin \theta_1\right) = A_{L1}f_{L1}(h-a_1-x_1/2) \left(\cos \theta_1 + r \frac{b \tan \theta_1}{b + 2h} \sin \theta_1\right).$$

If $\alpha = h/b$, this equation can be arranged to give

$$T = A_{L1}f_{L1}(h-a_1-x_1/2) \frac{1 + \frac{r}{1 + 2\alpha} \tan^2 \theta_1}{\frac{1}{\psi} + \tan \theta_1} \qquad \ldots \quad (5.4)$$

The inclination, θ_1, of the hinge will be such as to make the failure torque a minimum. If $dT/d\theta_1$, is equated to zero, it is found that

T is a minimum when,

$$\tan\theta_1 = -\frac{1}{\psi} + \sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{1+2\alpha}{r}} \qquad \ldots \qquad (5.5)$$

When this value is substituted into equation (5.4) the failure torque for a mode 1 failure is obtained as,

$$T_1 = A_{L1}f_{L1}(h-a_1-x_1/2)\,\frac{2r}{1+2\alpha}\left(\sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{1+2\alpha}{r}} - \frac{1}{\psi}\right) \qquad \ldots \qquad (5.6)$$

If M_u is the ultimate capacity of the member in simple flexure, i.e.

$$M_u = A_{L1}f_{L1}(h-a_1-x_1/2),$$

equation (5.6) reduces to

$$\frac{T_1}{M_u} = \frac{2r}{1+2\alpha}\left(\sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{1+2\alpha}{r}} - \frac{1}{\psi}\right) \qquad \ldots \qquad (5.7)$$

5.1(b) Mode 3

When the beam contains less longitudinal steel in the top than in the bottom, and is loaded predominantly in torsion, a mode 3 failure is possible.

In a mode 3 failure the compression zone forms along the bottom face of the beam (Figure 5.3) i.e. the face on which bending moment alone would cause tension. The analysis is very similar to that for mode 1, except that the bending moment now opposes the rotation occurring during failure, in this mode. Assuming that the cover to the top and bottom longitudinal steel is the same, equilibrium of forces about the failure axis yields the following equation

$$T\left(\sin\theta_3 - \frac{\cos\theta_3}{\psi}\right)$$

$$= A_{L3}f_{L3}(h-a_1-x_3/2)\left(\cos\theta_3 + r\,\frac{b\tan\theta_3}{b+2h}\sin\theta_3\right) \qquad \ldots \qquad (5.8)$$

If this expression is minimised with respect to Θ_3 the equation for the failure torque is obtained.

$$\frac{T_3}{A_{L3}f_{L3}\ (h-a_1-x_3/2)} = \frac{2r}{R(1+2\alpha)}\left(\sqrt{(\frac{1}{\psi})^2 + \frac{(1+2\alpha)R}{r}} + \frac{1}{\psi}\right) \quad \ldots \quad (5.9)$$

where,

$$R = \frac{A_{L3}f_{L3}}{A_{L1}f_{L1}}$$

For this mode to be critical A_{L3} must be less than A_{L1} then, x_3 will be smaller than x_1 (which has been taken equal to the depth of the compression in pure flexure). Thus it is conservative to assume that

$$h-a_1-x_3 = h-a_1-x_1$$

Equation (5.9) then becomes

$$\frac{T_3}{M_u} = \frac{2r}{1+2\alpha}\left(\sqrt{(\frac{1}{\psi})^2 + \frac{(1+2\alpha)R}{r}} + \frac{1}{\psi}\right) \quad \ldots \quad (5.10)$$

For a given ψ and known beam dimensions the torques T_1 and T_3 can be computed from equations (5.8) and (5.10). The smaller of these values will be the twisting moment at failure in bending and torsion.

5.1(c) Bending-Torsion Interaction

Although the equations given above can be used for the prediction of the failure load of a beam, the physical significance of these formulae is more apparent if they are recast in the form of interaction curves. Now,

$$\frac{T_1}{M_u} = \frac{2r}{1+2\alpha}\left(\sqrt{(\frac{1}{\psi})^2 + \frac{1+2\alpha}{r}} - \frac{1}{\psi}\right),$$

Then, after rearrangement,

$$(\frac{T_1}{M_u} + \frac{2r}{1+2\alpha}\cdot\frac{1}{\psi}) = \frac{2r}{1+2\alpha}\ \sqrt{(\frac{1}{\psi})^2 + \frac{1+2\alpha}{r}},$$

and squaring

$$\left(\frac{T_1}{M_u}\right)^2 + \frac{2T_1}{M_u} \cdot \frac{1}{\psi} \cdot \frac{2r}{1 + 2\alpha} = \frac{4r}{1 + 2\alpha}$$

If $(\frac{1}{\psi}) = 0$ (i.e. pure torsion) is substituted into this equation, an expression for T_o, the pure torsional strength (mode 1) is derived.

$$T_o = 2M_u \sqrt{\frac{r}{1 + 2\alpha}} \quad \ldots \quad (5.11)$$

Hence the following equation for the bending and torsion interaction is obtained.

$$\left(\frac{T}{T_o}\right)^2 + \frac{M}{M_u} = 1 \quad \ldots \quad (5.12)$$

It can thus be seen that, provided the beam fails in mode 1, the effect of torsion is to reduce the flexural capacity of a beam, and vice versa.

The formula for the strength in mode 3 can also be expressed in a similar form

$$\left(\frac{T}{T_o}\right)^2 = \frac{M}{M_u} + R \quad \ldots \quad (5.13)$$

In this case it can be seen that flexure tends to increase the torsional capacity. The above equations have been plotted in Figure 5.4 for various values of R.

From this figure it is evident that when R is unity, that is, for beams with equal areas of top and bottom reinforcement, the mode 1 equation will always be more critical than mode 3. In this case bending will decrease the torsional strength. When R is less than unity, in the region of high torsion a mode 3 failure will occur. In this case the effect of bending is to increase the torsional capacity up to a certain point, then to decrease the torsional strength.

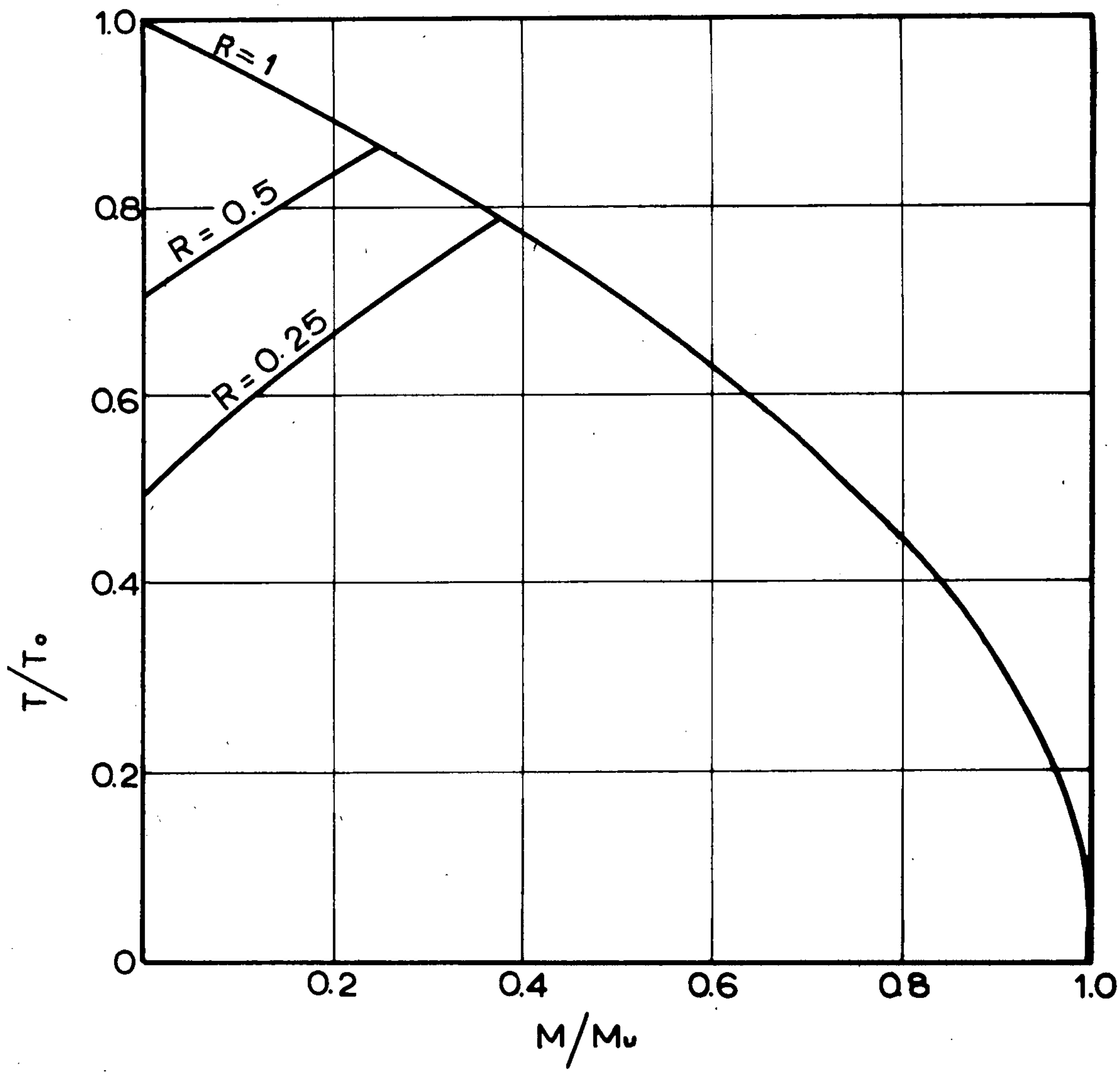


FIG. 5.4 EFFECT OF THE RATIO R ON THE SHAPE OF THE BENDING-TORSION INTERACTION CURVE.

5.2 SHEAR AND TORSION

In the presence of torsion combined with moderate shear force failure may take the form of a mode 2 mechanism or in certain cases a mode 3 mechanism. As both of these modes of failure require yielding of the longitudinal steel and take no account of the contribution to the failure load of the uncracked concrete, the resulting equations can not be satisfactorily applied to the case of high shear. Indeed it has been found that under predominately shear loading the failure behaviour resembles that of simple shear (Ref. 1.1). This type of failure has been designated as an "effective shear failure". In the following chapter an empirical formula for the strength of the beam in this mode of failure will be given. A more exact analysis has been made by Collins (Ref. 1.1).

5.2(a) Mode 2

In a mode 2 failure, the compression zone is located along one side of the beam (Figure 5.3) and the compression hinge is at a distance $x_2/2$ from the side face. The external bending moment has no component about this hinge axis, but the shear force does exert a moment about this axis.

The total moment of the external forces is

$$M_{ext.} = V(\frac{b}{2} - x_2/2) \sin \theta_2 + T \sin \theta_2$$

$$= T \{\delta(1 - \frac{x_2}{b}) \sin \theta_2 + \sin \theta_2\} \qquad \ldots \qquad (5.14)$$

where $\delta = Vb/2T$.

The internal moment is mainly provided by the longitudinal steel near the side face remote from the hinge and by the vertical legs of the stirrups on that face. If the same assumptions are made as in mode 1, this moment may be written as

$$M_{int.} = A_{L2}f_{L2}(b - a_2 - x_2/2)\cos\theta_2 + \frac{A_w f_w(h-2a_4)}{s}\ \frac{h\tan\theta_2}{h + 2b}(b-a_2-x_2/2)\sin\theta_2 \qquad \ldots \quad (5.15)$$

By a procedure similar to that used for mode 1, the expression for the failure torque for a mode 2 failure is found to be

$$T_2\ \{1 + \delta(1 - \frac{x_2}{b})\} = 2A_{L2}f_{L2}(b-a_2-x_2/2)\sqrt{\frac{A_w f_w(h-2a_4)}{A_{L2}f_{L2}s(1 + \frac{2}{\alpha})}} \qquad \ldots \quad (5.16)$$

For most beams we may take

$$A_{L2}f_{L2} = \tfrac{1}{2}(A_{L1}f_{L1} + A_{L3}f_{L3})$$

We then have

$$\frac{A_{L2}f_{L2}}{A_{L1}f_{L1}} = \frac{1 + R}{2}$$

Now, it is found that x_2 is small compared with x_1. Hence it is conservative to put

$$\frac{h - a_1 - x_1/2}{b - a_2 - x_2/2} = \frac{h - a_1}{b - a_2} = \beta$$

On the grounds of simplicity we may make the conservative assumption that,

$$\frac{h - 2a_4}{b - 2a_3} = \frac{h}{b} = \alpha$$

For the same reason it is safe, but not unduly conservative, to ignore the term x_2/b on the left hand side of equation (10). The expression for the failure torque in the second mode now takes the form

$$\frac{T_2}{M_u} = \frac{1}{1 + \delta}\cdot\frac{\alpha}{\beta}\sqrt{\frac{2(1 + R)r}{2 + \alpha}} \qquad \ldots \quad (5.17)$$

5.2(b) Mode 3

It was found in the previous section that a Mode 3 failure could occur if the beam contained less top than bottom longitudinal steel. The torsional strength of a beam failing in this mode was shown to increase rapidly with the ratio of moment to torque. However when shear is present the moment varies along the section and some difficulty may arise in estimating the effective moment on the failure mechanism. As all the specimens tested in bending torsion and shear were simply supported in bending and subjected to constant torque along the member an explicit formula for this situation will be derived.

The loading under consideration is shown in Figure 5.5. The effective moment M' can be seen from this figure to be,

$$M' = M_o \cdot \frac{b \tan \theta_3}{2a}$$

Now $$\psi = \frac{T}{M}$$

$$= \frac{T}{M_o} \cdot \frac{2a}{b \tan \theta_3}$$

But $$\frac{M_o}{a} = V$$

Therefore the expression for ψ reduces to,

$$\psi = \frac{1}{\tan \theta_3 \; \delta}$$

This value of ψ can be substituted in the equation for equilibrium of moments about the failure axis given for a normal mode 3 mechanism (equation 5.8). Following the same steps as before the equation for the failure torque is obtained

$$\frac{T_3}{M_u} = \frac{2}{1 - \delta}\sqrt{\frac{r.R}{1 + 2\alpha}} \qquad \ldots \qquad (5.18)$$

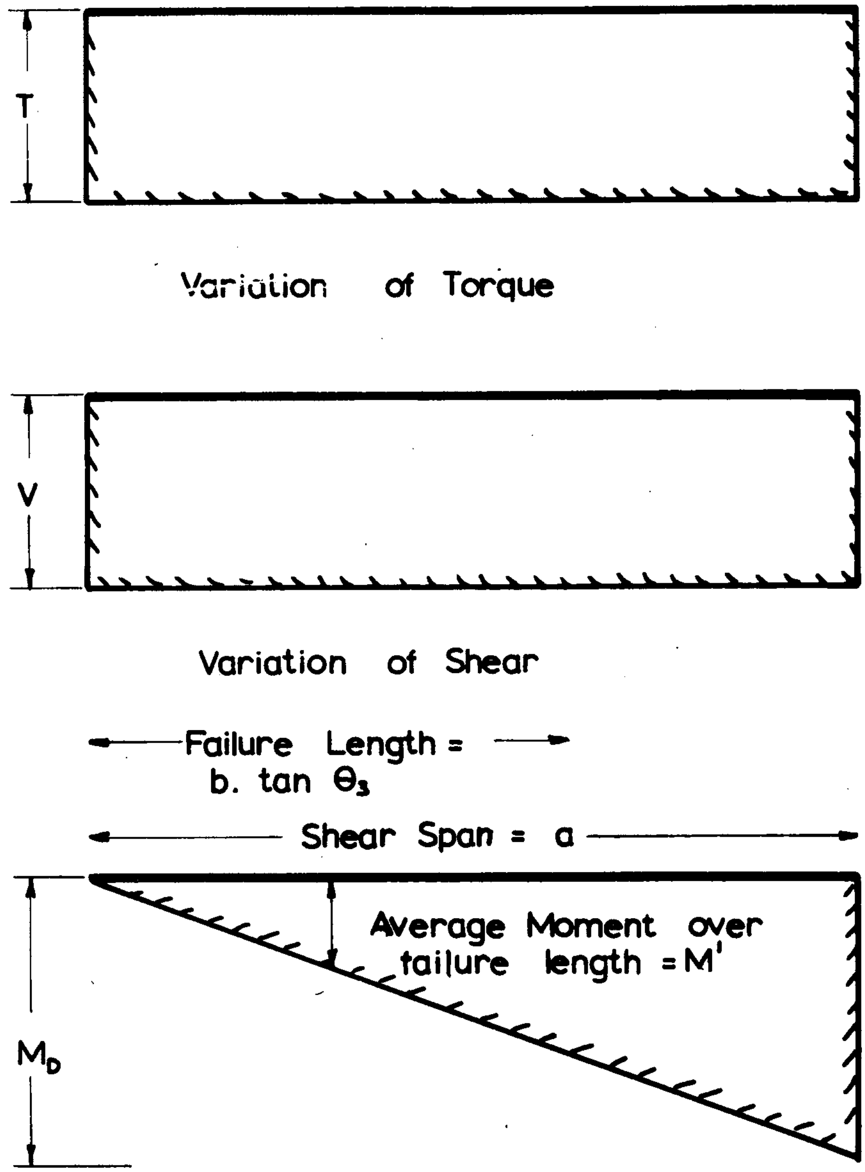


FIG. 5.5 DISTRIBUTION OF LOADS FOR SHEAR SPAN THIRD MODE FAILURE.

CHAPTER 6

EXPERIMENTAL VERIFICATION OF THE ULTIMATE STRENGTH THEORY FOR WEB REINFORCED BEAMS.

In this chapter the expressions derived in the previous section (Chapter 5) will be verified by comparison with experimental results. Test data which is used in this analysis includes both the author's test beams and the reported experimental results. Details of the author's tests are given in Chapter 3, and a summary of the available reported experiments is given in Appendix B and D. As several assumptions are made in the derivation of the ultimate strength equations, consideration will firstly be given to the effect of these assumptions on the range of validity of the theory. The ability of the theory to predict the interaction behaviour of beams will be tested and the overall accuracy assessed.

6.1 LIMITATIONS ON BEAM PROPORTIONS

A principal assumption in the derivation of the analysis equations was that all reinforcement crossed by the failure surface yielded. Under certain ratios of loads and certain arrangements of reinforcement failure may occur before both the longitudinal and transverse steel have yielded. For such beams it is necessary either to estimate the failure load by some other means or to limit the applicability of the theory to exclude these cases. Failure before yield of the longitudinal steel is possible if,

(i) The loading is such that a premature shear failure occurs,

(ii) The amount of longitudinal steel is out of proportion to the amount of transverse reinforcement.

(iii) The percentage of longitudinal reinforcement is excessive.

Additionally, failure may occur before yielding of the transverse reinforcement if the overall amount of steel is enough to cause general crushing of the concrete.

6.1(a) Effective Shear Failure

Under predominantly shear loading it was found by Collins that the failure behaviour of a reinforced member closely resembled the behaviour of similar specimens in pure shear. It would seem more satisfactory, in this case, to relate the ultimate strength to the shear capacity of the beam in simple flexure.

An examination has been made of the available experimental results for those beams which might be expected to fail in shear and torsion. From these results it was concluded that the simple equation given in section 4.5 for the case of longitudinally reinforced beams could also be applied to web reinforced members.

$$\text{i.e.} \quad V + 1.6T/b = V_{eff} = V_o$$

$$\text{or} \quad T_{eff} = \frac{bV_o}{1.6 + 2\delta} \qquad \ldots \qquad (6.1)$$

Equation (6.1), in conjunction with the equations given in Chapter 5, can be used to predict the ultimate strength of a member loaded in combined torsion bending and shear. A comparison of the predictions of these equations with the experimental results will be given later in this chapter.

6.1(b) Ratio of Transverse to Longitudinal Steel.

As tests in which the strains of the steel have been measured (6.1,6.2) show that for low values of the parameter r the longitudinal steel may not yield, an investigation has been made of the range for r for which the theory set out above would hold. As this range would depend on the factors ψ and α a parameter r_o incorporating these factors was introduced. The value of the parameter r_o corresponds to a design value of the ratio r for given values of ψ and α. The value of r_o may be calculated from the equation below:

$$r_o = \frac{1}{4 + \dfrac{4}{\psi\sqrt{1+2\alpha}}} \qquad \ldots (6.2)$$

The derivation of this equation is given in Appendix C.

In Figure 6.1 the parameter T_{exp}/T_{theor} which is a measure of the accuracy of the theory, is plotted against r/r_o where r is the actual value of r for the test beam and r_o is the optimum value of r as given by equation (6.1) above.

In constructing this figure use has been made of test results published in the literature. Figure 6.1 does not include results of tests where failure may have been initiated by crushing of the concrete (see below), or where a shear failure (see equation (6.1)) may have occured..

It will be noticed that for low values of r/r_o there is a wide scatter of experimental points, but the theory is still generally conservative. In this range, corresponding to beams with relatively small amounts of transverse steel, the idealised modes of failure are no longer applicable. Factors ignored in the observations above, such as tensile stresses in the concrete and dowel forces exerted by the steel, are now of considerable

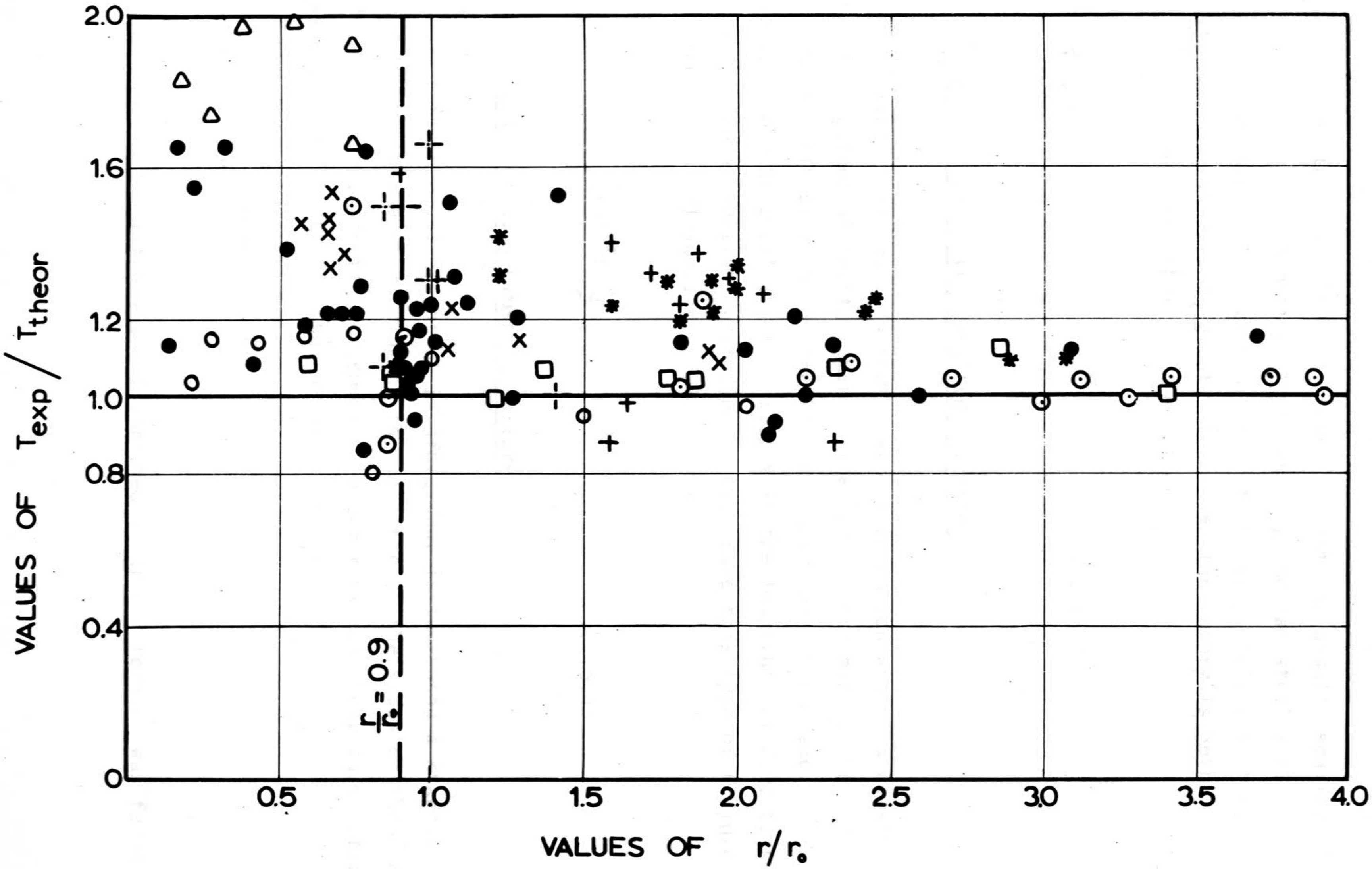


FIG. 6.1 THE EFFECT OF THE PARAMETER r/r_o ON THE ACCURACY OF THE THEORY.

importance. As the error thus introduced offset the usually smaller error involved in the assumptions that the longitudinal steel yields, the analysis equations still give usable, if not completely reliable, results. For values of r/r_o greater than 0.9, the theory is both consistent and accurate, that is,

$$\frac{r}{r_o} \nless 0.9 \qquad \ldots (6.3)$$

6.1(c) Excessive Longitudinal Steel

Lessig (6.4) reported that test specimens containing excessive amounts of longitudinal reinforcement failed prior to yield of the reinforcement, with crushing of the concrete near the top surface. Consideration of the equilibrium of forces in the longitudinal direction suggests that an appropriate criterion to eliminate this type of failure would be the normal flexural criterion.

$$\frac{(A_{L1} - A_{L3})\ f_L}{bdf'_c} \ngtr 0.4 \qquad \ldots (6.4)$$

6.1(d) General Crushing Failure

If the overall amount of reinforcement is increased a stage is eventually reached where crushing of the concrete on the sides of the beam precedes yielding of the reinforcement. Under flexural shear this limit is expressed in terms of the nominal shear stress

$$\frac{V}{bd} \ngtr 8 \sqrt{f'_c}$$

The limit for the case of combined shear and torsion was found to be

$$\frac{V_{eff}}{bd} = \frac{V + 1.6\ T/b}{bd} \not> 8\sqrt{f'_c} \qquad \ldots (6.5)$$

The effect of this limit is shown in Figure 6.2 where T_{exp}/T_{theor} has been plotted against $\frac{V_{eff}}{bd\ f'_c}$. It can be seen that for high values of the nominal "shear" stress, that is for excessively reinforced beams, the theory becomes unconservative. For such beams the assumption that the reinforcement yields, leads to an overestimate of the failure load. Within the limit the theory is accurate. If it is required to calculate the strength of a specimen in which the above limit is exceeded, it is suggested that the excess reinforcement be ignored in the analysis.

6.2 INTERACTION OF BENDING AND TORSION.

A good test of the accuracy of any theory is its ability to explain the bending and torsion interaction behaviour. In the previous chapter the following expressions were obtained for the theoretical interaction behaviour implicit in the analysis equations.

Mode 1 :

$$\left(\frac{T}{T_o}\right)^2 + \frac{M}{M_u} = 1 \qquad \ldots (6.6)$$

Mode 3 :

$$\left(\frac{T}{T_o}\right)^2 = \frac{M}{M_u} + R \qquad \ldots (6.7)$$

It was found that equation (6.7) only applies to beams which contain less top longitudinal steel than bottom steel, (i.e. $R < 1$) and even then only

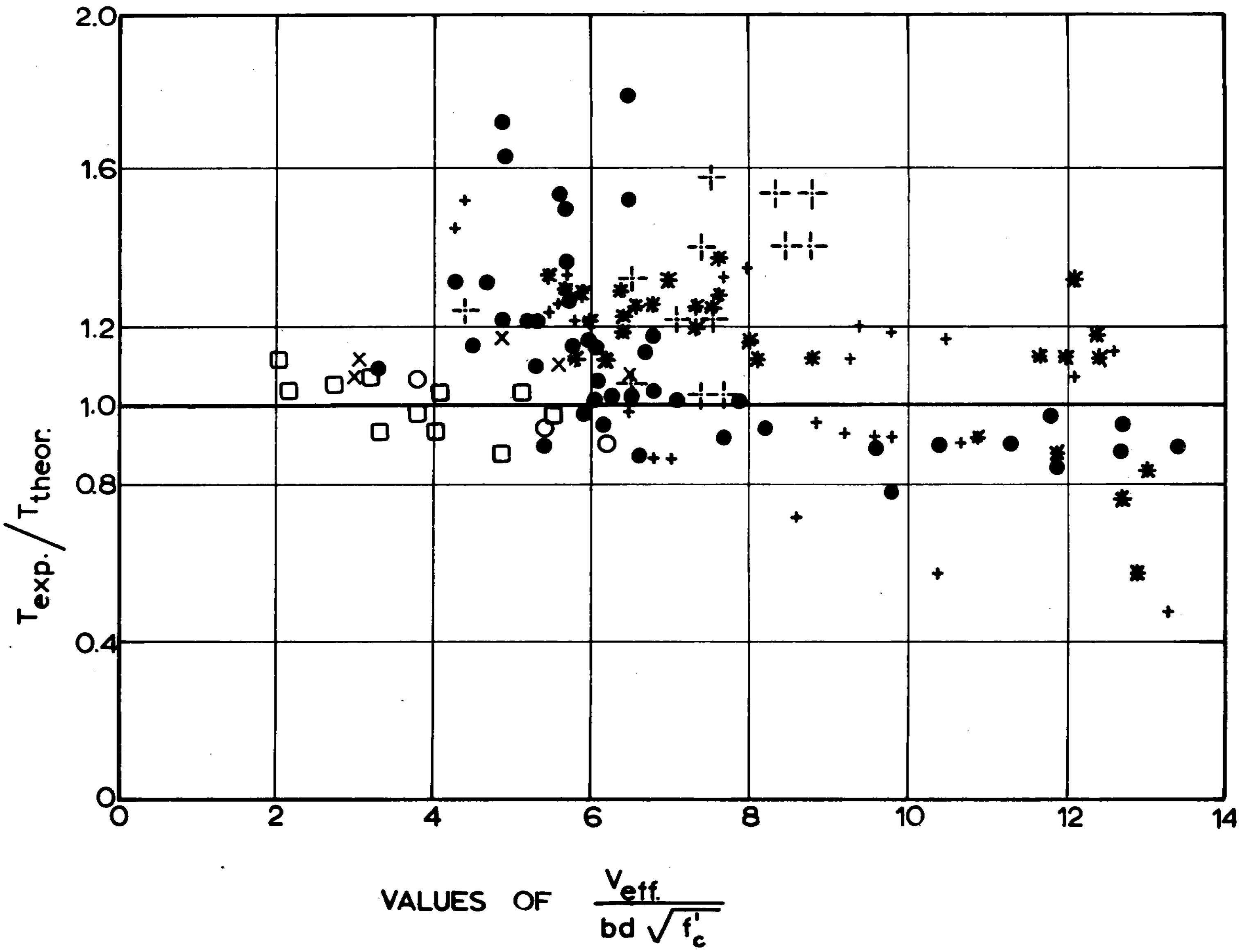


FIG. 6.2 THE EFFECT OF THE NOMINAL SHEAR STRESS ON THE ACCURACY OF THE THEORY.

to beams loaded with high ratios of torsion to bending. Moreover this equation predicted that the torsional capacity would be increased by the addition of bending moment. To test the above equation a series of test beams which contained more tension than compression steel were tested. The results of these tests are compared with the theoretical expression in Figure 6.3. It can be seen that the expressions agree well with the experimental trends. The majority of tests are governed by the mode 1 equation. The results of these tests are shown in Figure 6.4. In this case the beams that were tested predominantly in torsion were all reinforced with equal areas of top and bottom reinforcement, hence the equation for mode 3 was never critical. The results shown in Figure 6.4 show a marked interaction between torsion and bending, substantial reductions in the flexural capacity being produced by the effect of torsion. Again the theoretical interaction curve closely follows the experimental data.

6.3 ACCURACY OF ANALYSIS EQUATIONS

Full advantage has been taken of the large number of test results that have been reported in the literature. A summary of this data is presented in Appendix B and D. Frequently in reinforced concrete research theories that give good agreement with the investigators own tests, subsequently show discrepancies when other test data are analysed. In the author's analysis of the experimental results no distinction will be made between the results of the tests of this investigation and the results which have been reported in the literature. The range of section parameters encompassed in the available test data is given below:

Parameter	Minimum	Maximum
h/b	1.00	2.36
f'_c	680 p.s.i.	8,500 p.s.i.
$\frac{(p-p')f_y}{f'_c}$	0	1.00
ψ	0.025	∞
δ	0.0	4.11

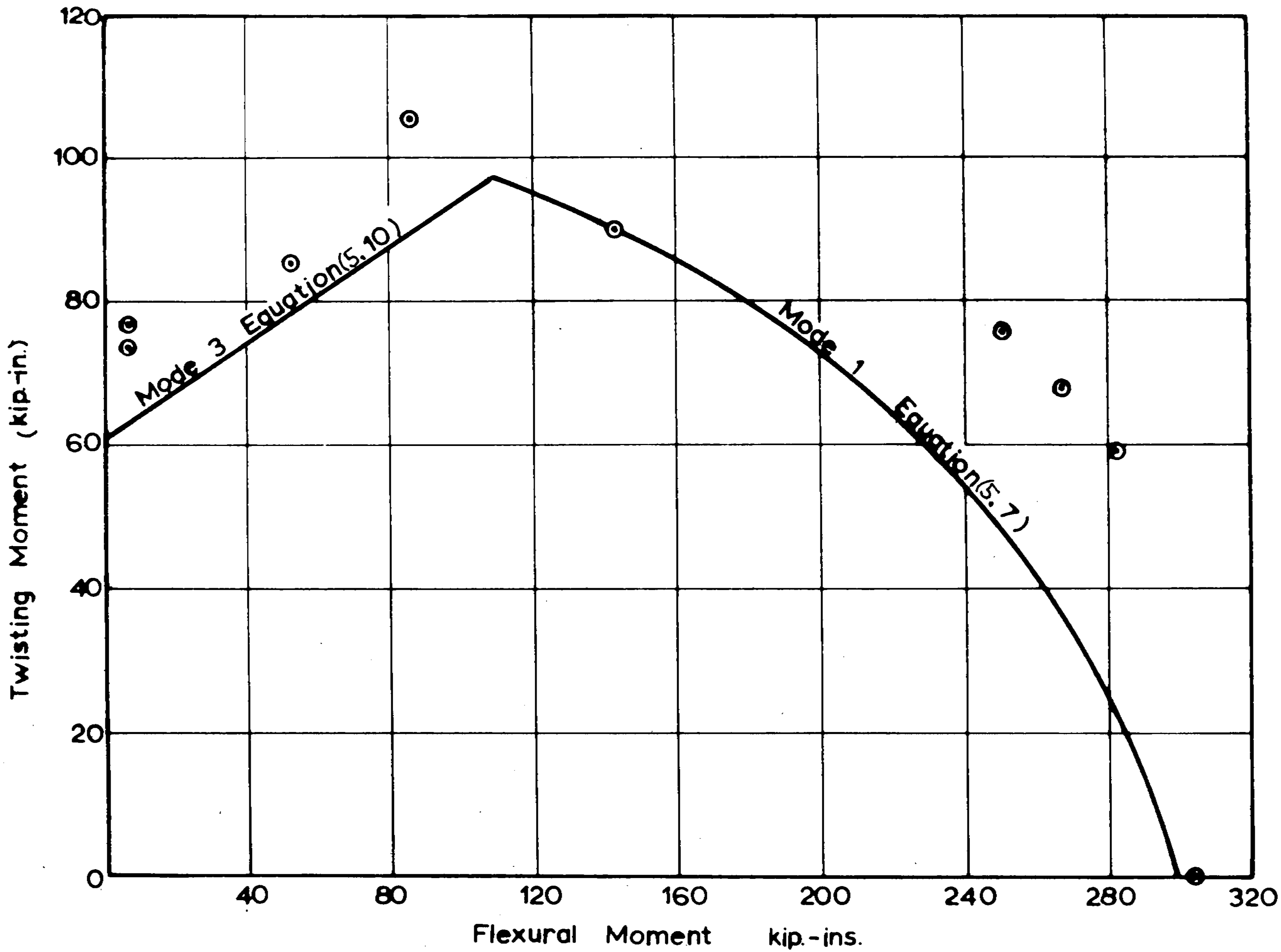


FIG. 6.3 INTERACTION CURVE FOR RU SERIES.

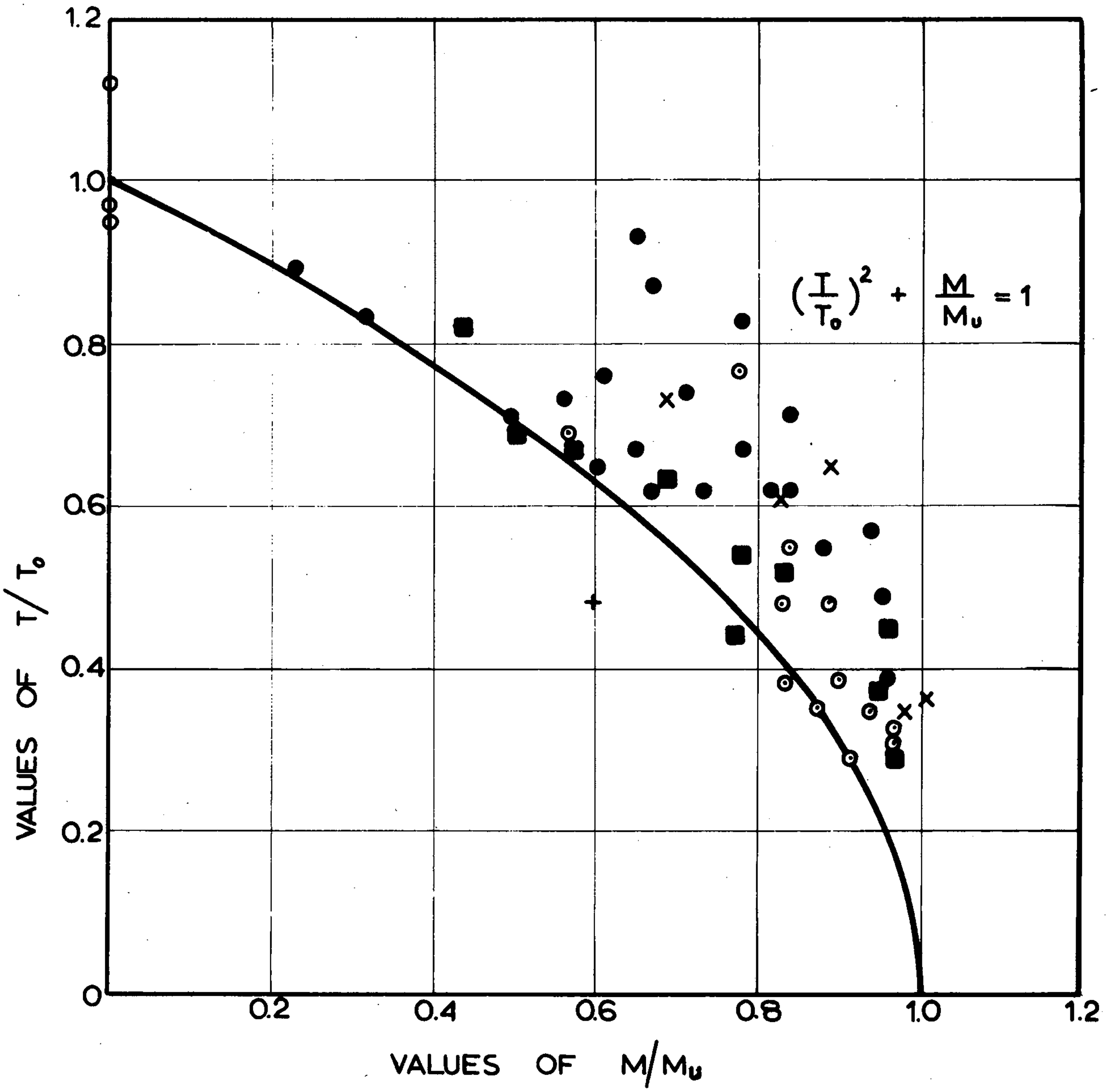


FIG. 6.4 BENDING TORSION INTERACTION FOR WEB REINFORCED BEAMS

6.3(a) Analysis Method

In Chapter 5 and in section 6.1(a) the following equations for the failure loads of a web reinforced beams were given,

$$\frac{T_1}{M_u} = \frac{2r}{1 + 2\alpha}\left(\sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{1 + 2\alpha}{r}} - \frac{1}{\psi}\right) \qquad \ldots \quad (6.8)$$

$$\frac{T_3}{M_u} = \frac{2r}{1 + 2\alpha}\left(\sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{(1 + 2\alpha)R}{r}} + \frac{1}{\psi}\right) \qquad \ldots \quad (6.9)$$

or

$$\frac{T_3}{M_u} = \frac{2}{1 - \delta}\sqrt{\frac{r.R}{1 + 2\alpha}} \qquad \ldots \quad (6.10)$$

$$\frac{T_2}{M_u} = \frac{1}{1 + \delta}\,\frac{\alpha}{\beta}\sqrt{\frac{2(1 + R)r}{2 + \alpha}} \qquad \ldots \quad (6.11)$$

$$T_{eff} = \frac{bV_o}{1.6 \pm 2\delta} \qquad \ldots \quad (6.1)$$

Earlier in this chapter graphs were presented for the effect of various parameters on the accuracy of the theory. Examination of these figures shows that within the restrictions imposed on the theory the following parameters have no appreciable effect on the accuracy of the proposed method; (i) the ratio of longitudinal to transverse steel, (Figure 6.1) and (ii) the proportion of shear and torsional loading to the size of the section (Figure 6.2)

Table 6.1 has been prepared to investigate the accuracy of the theory for beams satisfying the restrictions set out in equations 6.3,6.4

TABLE 6.1

A COMPARISON OF EXPERIMENTAL RESULTS WITH THE PROPOSED THEORY FOR WEB REINFORCED BEAMS

PART 1 - PURE TORSION

Investigator	Beam	Torque kip. in.	Moment kip. in.	Shear kips	$\frac{T_{exp}}{T_{theor}}$	Mode
Ernst	3TR15	61.7	0.0	0.0	1.12	1
	3TR30	76.0	0.0	0.0	0.97	1
	4TR30	85.0	0.0	0.0	0.95	1
Evans	HB1	44.1	0.0	0.0	1.15	3
Lessig	BK1	121.0	0.0	0.0	0.98	2
	BK1A	104.0	0.0	0.0	0.87	2
				Mean	1.01	
				Standard Deviation	10 °/o	
				No. of Tests	6	

PART 2. BENDING AND TORSION

Investigator	Beam	Torque kip. in.	Moment kip. in.	Shear kips.	$\frac{T_{exp}}{T_{theor}}$	Mode
This Investigation	R4.20	59.9	331.0	0.0	1.15	1
	R4.24	56.5	264.0	0.0	0.96	1
	R3.20	50.7	252.0	0.0	1.08	1
	R3.24	53.7	230.0	0.0	1.03	1
	R3.30	61.6	207.0	0.0	1.02	1
	R2.24	44.2	205.0	0.0	1.24	1
	R2.30	49.7	176.0	0.0	1.18	1
	R2.38	53.4	138.0	0.0	1.06	1
	R1.30	41.8	146.0	0.0	1.07	1
	RE1	81.4	6.3	0.0	0.88	2
	RE2	83.5	32.0	0.0	1.01	1
	RE3	81.5	45.0	0.0	1.01	1
	RE4	74.6	84.4	0.0	1.12	1
	RE5	66.0	108.2	0.0	1.16	1
	RE4*	38.0	134.0	0.0	1.10	1
	RU3	105.0	84.0	0.0	1.26	3
	RU3A	89.4	149.3	0.0	1.00	1
	RU2	84.9	51.1	0.0	1.11	3
	RU5	75.4	249.7	0.0	1.17	1
	RU5A	68.3	266.8	0.0	1.14	1
	RU6	59.1	281.2	0.0	1.15	1
	36T4	62.6	240.4	0.0	1.21	1
	36T4C	94.1	61.1	0.0	1.23	3
	36T5.5	85.9	173.4	0.0	1.32	1
	77T5	91.6	262.4	0.0	1.31	1
	77T4	107.6	223.4	0.0	1.26	1
	24T3	70.8	46.6	0.0	0.92	2
Gesund Schuette Buchanan and Gray	1	79.0	79.0	0.0	1.07	1
	2	102.0	102.0	0.0	1.01	1
	3	61.0	122.0	0.0	1.06	1
	4	67.0	134.0	0.0	0.96	1
	5	49.0	147.0	0.0	1.08	1
	6	56.0	168.0	0.0	1.08	1
	7	43.0	173.0	0.0	1.14	1
	8	44.0	176.0	0.0	1.05	1
	11	68.0	138.0	0.0	0.98	1
	12	53.0	213.0	0.0	1.06	1

PART 2. BENDING AND TORSION (contd.)

Investigator	Beam	Torque kip. in.	Moment kip. in.	Shear kips.	$\frac{T_{exp}}{T_{theor}}$	Mode
Evans and Sarkar	HB2	33.9	66.8	0.0	1.25	1
	HB3	20.4	75.3	0.0	1.10	1
	HB4	15.7	81.6	0.0	1.05	1
	HB5	13.2	81.5	0.0	1.06	1
	HB8	21.4	79.6	0.0	1.05	1
	HB9	18.3	85.1	0.0	0.98	1
	HB10	17.3	91.3	0.0	0.99	1
	HB11	14.1	94.0	0.0	0.99	1
	HB14	41.7	82.1	0.0	1.03	1
	HB15	29.9	111.0	0.0	1.11	1
	HB16	23.5	129.0	0.0	1.05	1
	HB17	19.4	137.0	0.0	1.06	1
Chinenkov	B28 0.1	48.6	486.0	0.0	1.12	1
	B28 0.1A	46.9	469.0	0.0	1.09	1
	B28 0.2	83.4	417.0	0.0	1.14	1
	B28 0.2A	83.4	417.0	0.0	1.23	1
	B28 0.4F	139.0	347.0	0.0	1.15	1
Lessig	BU6	69.5	173.6	0.0	0.87	1
			Mean		1.10	
			Standard Deviation		9 °/o	
			No. of Tests		55	

PART 3. SHEAR AND TORSION

Investigator	Beam	Torque kip. in.	Moment kip. in.	Shear kips	$\frac{T_{exp}}{T_{theor}}$	Mode
This Investigation	RU4	85.5	145.0	4.13	1.21	3V
	R1.30A	42.6	97.1	4.13	0.89	VEF
	R3.20B	59.0	78.9	3.41	1.19	3V
Collins	V3	16.9	685.0	27.80	1.65	VEF
	V6	24.8	668.0	27.20	1.72	VEF
	U2	43.9	689.0	28.00	1.53	VEF
	U3*	66.2	720.0	29.30	1.80	VEF
	T4	53.0	523.0	21.40	1.54	2
	T5	63.4	432.0	17.70	1.52	2
	T6*	29.4	584.0	23.90	1.36	VEF
Yudin	6	4.6	45.7	2.32	1.01	1
	10	5.9	59.0	3.00	1.31	1
	11	14.3	71.7	3.65	1.66	1
	12	11.1	55.7	2.83	1.29	1
	13	11.1	55.7	2.83	1.29	1
	22	6.5	32.6	1.66	1.28	1
Lessig	BПI5	156.0	416.0	15.56	1.40	2
	BПI5A	151.0	416.0	15.52	1.33	1
	BПI6	92.0	156.0	4.07	1.31	1
	BПI6A	83.4	156.0	4.16	1.27	1
	BПI7A	90.4	313.0	8.06	1.37	1
	BПI7	83.4	278.0	7.16	1.24	1
	BПI9	78.0	313.0	4.63	1.50	1
	BП 19A	79.0	313.0	4.71	1.59	1

PART 3. SHEAR AND TORSION (contd.)

Investigator	Beam	Torque kip. in.	Moment kip. in.	Shear kips	$\frac{T_{exp}}{T_{theor}}$	Mode
	B8 0.1	52.0	520.0	12.52	1.11	1
	B8 0.1A	55.5	555.0	13.36	1.11	1
	B8 0.2A	97.0	486.0	11.69	1.20	1
	B8 0.4A	139.0	347.0	8.38	1.16	VEF
	B7 0.2	93.8	468.0	11.30	1.31	VEF
	B7 0.2A	90.2	451.0	10.87	1.22	VEF
	B10 0.2	104.0	521.0	12.53	1.26	1
	B10 0.2A	104.0	521.0	12.53	1.23	1
	B1	90.3	452.0	10.82	1.33	1
Lyalin	B1A	90.3	452.0	12.75	1.29	1
	B2	139.0	694.0	16.65	1.29	1
	B2A	139.0	694.0	16.65	1.22	1
	B3	194.0	486.0	17.48	1.25	VEF
	B3A	194.0	486.0	17.48	1.24	VEF
	B5	194.0	972.0	23.24	1.22	1
	B5A	194.0	972.0	23.24	1.31	1
	B6	167.0	833.0	20.19	1.31	1
	B6A	181.0	903.0	21.88	1.43	1
			Mean		1.33	
			Standard Deviation		13 °/o	
			No. of Tests		42	

Summary of All Tests

Mean	1.18
Standard Deviation	15 °/o
No. of Tests	103

and 6.5. It should be noted that these restrictions are not very onerous. The limits given in equations 6.4 and 6.5 are merely the usual restrictions encountered in flexure and shear design. The limit on the ratio of transverse to longitudinal steel although of some importance in analysing test results does not represent any difficulty in design as will be shown in a later chapter.

In this analysis when the failure loads is governed by equation 6.8 the mode has been designated as 1, similarly modes 3, 3V, 2 and VEF refer to equations 6.9, 6.10, 6.11 and 6.1. Table 6.1 has been subdivided into three sections; pure torsion, bending and torsion, and bending, torsion and shear.

For the pure torsion test results contained in Table 6.1 good agreement between the experimental results and predicted capacities is obtained. The values of T_{exp}/T_{theor} have a mean of 1.01 and a standard deviation of 10%.

Examination of Table 6.1 shows that the proposed theory accurately predicts the ultimate strength of members loaded in combined bending and torsion. For this case T_{exp}/T_{theor} has a mean of 1.10and a standard deviation of 9%.

The accuracy of the proposed theory for shear and torsion is to a large extent dependant on the accuracy of the A.C.I. equations for pure shear. An anlysis of 166 beam tests reported in reference 6.5 yielded a mean V_{exp}/V_{ACI} of 1.44 $\pm$ 24% for beams with web reinforcement loaded in shear and flexure. The comparison between the theory and the experimental results for beams loaded in combined torsion, bending and shear is given in Table 6.1. It can be seen from this table that the theory is somewhat conservative for this case, the value of T_{exp}/T_{theor} having a mean of 1.33 and a standard deviation of 13%. When a more accurate shear design method becomes accepted, the accuracy of the theory for shear combined with torsion and flexure could be improved.

The overall accuracy of the theory for all the results in Table 6.1 is demonstrated in Figure 6.5. It may be concluded from this frequency distribution that the proposed theory is sufficiently accurate for safe and efficient design of structural members to resist combined torsion, bending and shear loads.

As occasions will probably arise when it is necessary to estimate the failure load of a member which has proportions outside the limitations on the theory, an investigation was made into the accuracy of the theory when no restrictions are imposed. Full details of this analysis is given in Appendix D. The results of this comparison of the theoretical and experimental failure loads is summarised in the frequency distribution shown in Figure 6.6. It can be seen from this figure that the theory is still reasonably accurate over a wide range of section proportions.

6,3(b) Simplified Analysis Procedure

In Chapter 7 the analysis equations will be rephrased into a form more suitable for rapid design. Considerable simplification of the design procedure results if the two equations relating to shear and torsion can be replaced by only one equation. In fact the effective shear formula was developed with this possibility in mind. With very little loss of accuracy the equation for the strength in mode 2 can be ignored in the analysis of the strength of a member. In Table 6.2 the results shown in Table 6.1 have been analysed using only equations 6.8, 6.10, 6.9 and 6.1.

A study of this table shows that the correlation between theoretical and experimental results is still very good. The mean

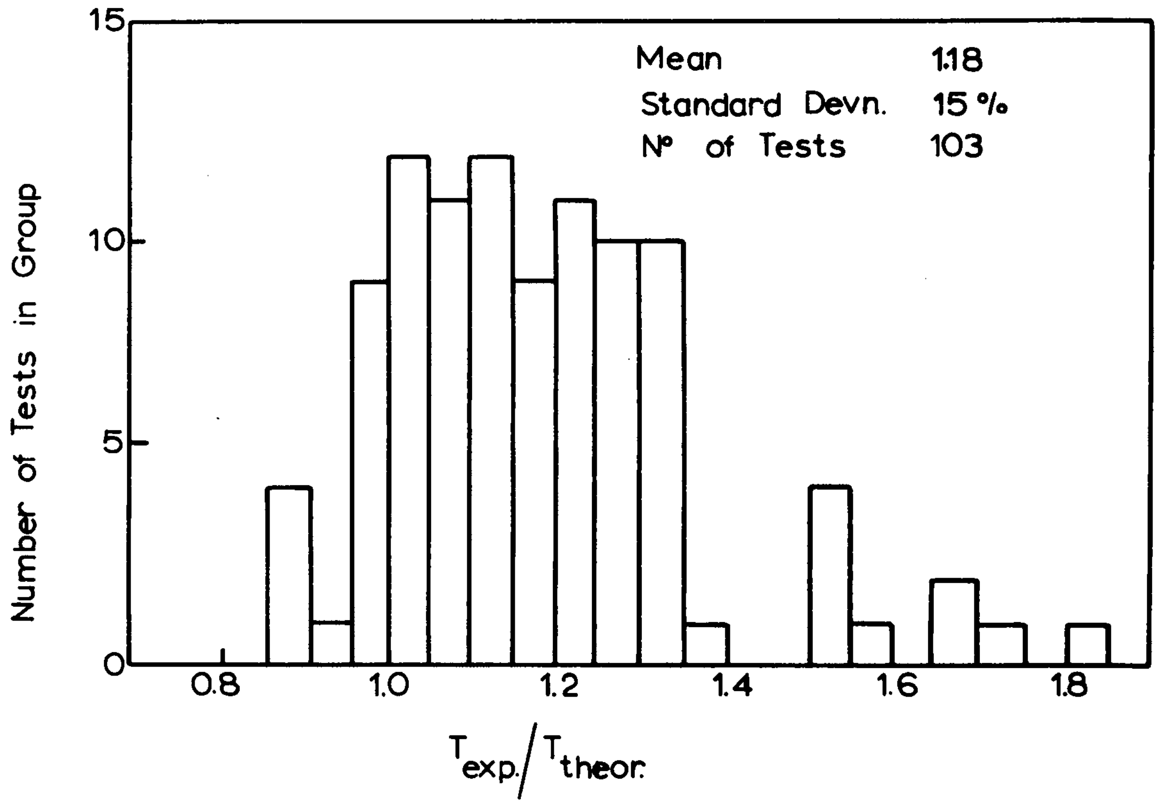


FIG. 6.5 FREQUENCY HISTOGRAM FOR THE ACCURACY OF THE THEORY FOR WEB REINFORCED BEAMS WITHIN RESTRICTIONS

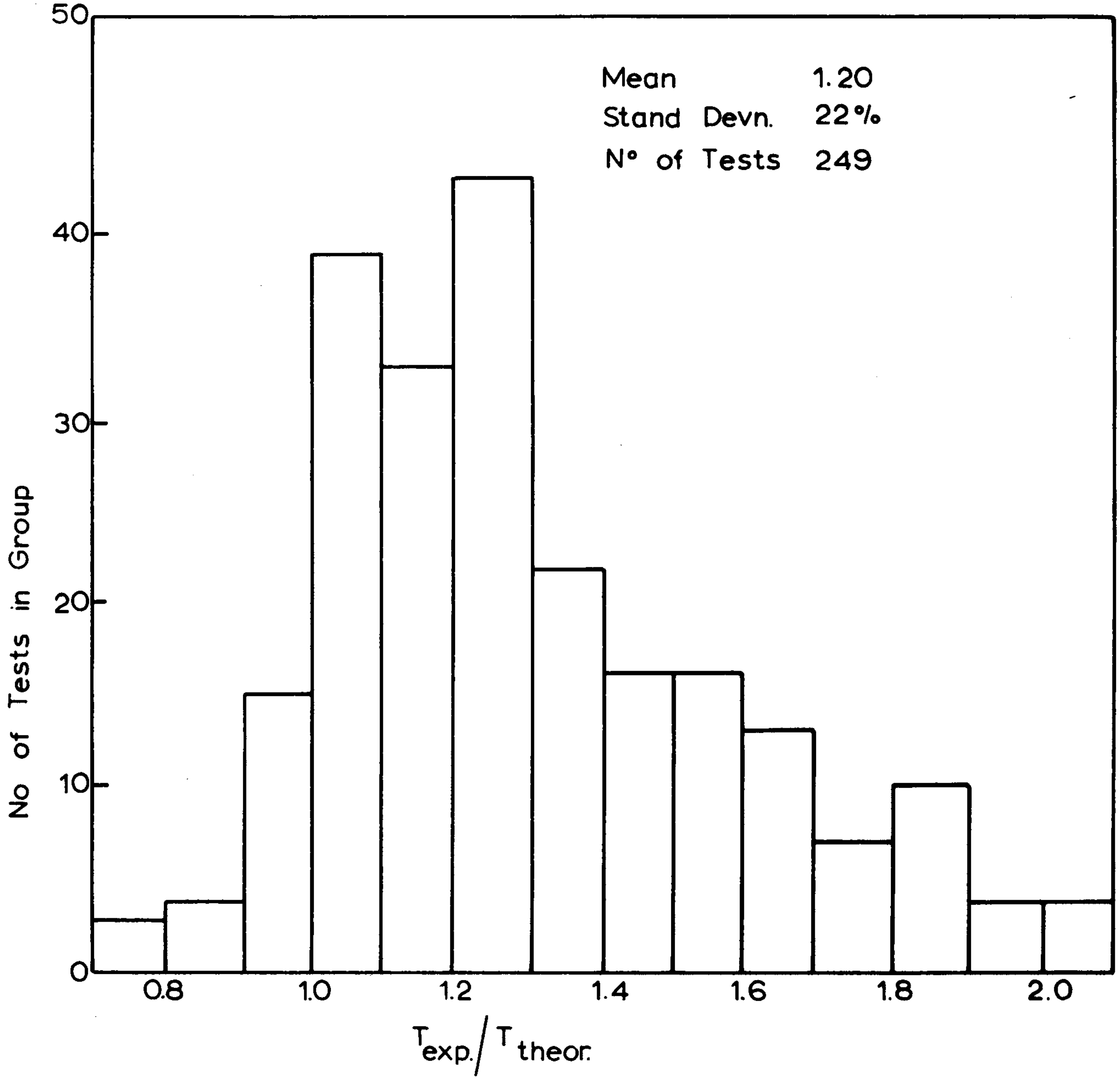


FIG. 6.6 FREQUENCY HISTOGRAM FOR THE ACCURACY OF THE THEORY FOR WEB REINFORCED BEAMS.
NO RESTRICTIONS.

TABLE 6.2

A COMPARISON OF THE DESIGN THEORY WITH TEST RESULTS FOR WEB REINFORCED BEAMS LOAD IN TORSION, BENDING AND SHEAR

Investigator	Beam	$\frac{T_{exp}}{T_{theor}}$	Mode
This Investigation	R4.20	1.15	1
	R4.24	0.97	VEF
	R3.20	1.08	1
	R3.24	1.02	1
	R3.30	1.02	1
	R2.24	1.24	1
	R2.30	1.18	1
	R2.38	1.06	1
	R1.30	1.07	1
	R1.30A	0.89	VEF
	R3.20B	1.19	3V
	RE1	0.85	1
	RE2	1.01	1
	RE3	1.01	1
	RE4	1.12	1
	RE5	1.16	1
	RE4*	1.10	1
	RU2	1.11	3
	RU3	1.26	3
	RU3A	1.00	1
	RU4	1.21	3V
	RU5	1.17	1
	RU5A	1.14	1
	4U6	1.15	1
	36T4	1.21	1
	36T4C	1.23	3
	36T5.5	1.32	1
	77T5	1.31	1
	77T4	1.26	1
	24T3	0.90	1
Chinenkov	B28 0.1	1.12	1
	B28 0.1A	1.09	1
	B28 0.2	1.14	1
	B28 0.2A	1.23	1
	B28 0.4F	1.15	1

Investigator	Beam	$\frac{T_{exp}}{T_{theor}}$	Mode
Collins	V3	1.65	VEF
	V6	1.72	VEF
	U2	1.53	VEF
	U3*	1.80	VEF
	T4	1.50	VEF
	T5	1.45	VEF
	T6*	1.36	VEF
Ernst	3TR15	1.12	1
	3TR30	0.97	1
	4TR30	0.95	1
Gesund	1	1.07	1
	2	1.01	1
	3	1.06	1
	4	0.96	1
	5	1.08	1
	6	1.08	1
	7	1.14	1
	8	1.05	1
	10	1.04	1
	11	0.98	1
	12	1.06	1
Lessig	BK1	0.97	1
	BK1A	0.84	1
	BU6	0.87	1
	BIII5	1.35	1
	BIII5A	1.33	1
	BIII6	1.31	1
	BIII6A	1.27	1
	BIII7A	1.37	1
	BIII7	1.24	1
	BIII9	1.50	1
	BIII9A	1.59	1

TABLE 6.2 (contd.)

Investigator	Beam	$\frac{T_{exp}}{T_{theor}}$	Mode
Lyalin	B8 0.2A	1.20	1
	B8 0.1	1.11	1
	B8 0.1A	1.11	1
	B8 0.4A	1.16	VEF
	B7 0.2	1.31	VEF
	B7 0.2A	1.22	VEF
	B10 0.2	1.26	1
	B10 0.2A	1.23	1
	B1	1.33	1
	B1A	1.29	1
	B2	1.29	1
	B2A	1.22	1
	B3	1.25	VEF
	B3A	1.24	VEF
	B5	1.22	1
	B5A	1.31	1
	B6	1.31	1
	B6A	1.43	1
Yudin	6	1.01	1
	10	1.31	1
	11	1.66	1
	12	1.29	1
	13	1.29	1

Investigator	Beam	$\frac{T_{exp}}{T_{theor}}$	Mode
Evans	HB1	1.15	3
	HB2	1.25	1
	HB3	1.10	1
	HB4	1.05	1
	HB5	1.06	1
	HB8	1.05	1
	HB9	0.98	1
	HB10	0.99	1
	HB11	0.99	1
	HB14	1.03	1
	HB15	1.11	1
	HB16	1.05	1
	HB17	1.06	1
Mean		1.18	
Standard Deviation		15 °/o	
No. of Tests		103	

T_{exp}/T_{theor} is 1.18 $\pm$ 15%. In view of this result the mode 2 mechanism will not be considered in the developement of the design equations.

CHAPTER 7

DESIGN OF BEAMS TO RESIST TORSION COMBINED WITH BENDING AND SHEAR.

The ultimate strength theories presented in Chapters 4 and 5 can be used to design members sustaining torsion combined with bending and shear. For beams with web reinforcement it is first necessary to rearrange the equations into a suitable form. The design method will then be outlined and illustrated with a design example.

7.1 DESIGN EQUATIONS FOR BEAMS WITH WEB REINFORCEMENT.

The given bending moment and torque can be resisted by many different beams. As usual in design, a choice is first made of some parameters, and the remaining dimensions are then determined to satisfy the basic equations. The design is commenced by an arbitary choice of ($\alpha = \frac{h}{b}$).

The parameter r may still be chosen from within a wide range of values. This means, physically, that there is a good deal of choice as to how the load is shared between the longitudinal and transverse steel. The choice of r is limited only by the fact that both steels must yield at failure. The design can be considerably simplified if r is chosen as an "optium value" r_o, which gives something approaching a minimum volume of steel (see Appendix C). This value is given by

$$r_o = \frac{1}{4 + 4/\psi\ (\sqrt{1 + 2\alpha})} \qquad \ldots \qquad (7.1)$$

If this expression is substituted for r in Equation 5.7 we obtain

$$\frac{T_1}{M_u} = \frac{2\ r_o}{1 + 2\alpha}\left(\sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{1 + 2\alpha}{r_o}} - \frac{1}{\psi}\right)$$

$$\frac{T_1}{M_u} = \frac{2r_o}{1 + 2\alpha}\left(\sqrt{\left(\frac{1}{\psi}\right)^2 + (1 + 2\alpha)\left(4 + \frac{4}{\psi\sqrt{1 + 2\alpha}}\right)} - \frac{1}{\psi}\right)$$

$$= \frac{2r_o}{1 + 2\alpha}\left(\sqrt{\left(\frac{1}{\psi} + 2\sqrt{1 + 2\alpha}\right)^2} - \frac{1}{\psi}\right)$$

$$= \frac{4r_o}{\sqrt{1 + 2\alpha}} \qquad \ldots \qquad (7.2)$$

The beam is designed primarily against failure in mode 1, and then checked for other modes of failure. We require first, therefore, that $T_1 = T$.

$$\text{i.e.}\quad M_u \frac{4r_o}{\sqrt{1 + 2\alpha}} = T$$

$$M_u = \frac{T\sqrt{1 + 2\alpha}}{4r_o}$$

When the expression for r_o as given in equation 7.1 is substituted in the above formula we obtain

$$M_u = T(\sqrt{1 + 2\alpha})(1 + \frac{1}{\psi\sqrt{1 + 2\alpha}})$$

$$= M + T\sqrt{1 + 2\alpha}$$

$$= M + T' \qquad \ldots \quad (7.3)$$

where $T' = T\sqrt{1+2\alpha}$

Thus, from the given values of M and T and the chosen value of α we find the equivalent ultimate moment M_u. The beam dimensions and A_{L1} are now designed for this M_u from the usual flexural equation.

$$M_u = A_{L1} f_{L1} d \left(1-0.59 \frac{A_{L1} f_{L1}}{f'_c \; bd}\right) \qquad \ldots \; (7.4)$$

with the proviso that $pf_{L1}/f'_c \leqslant 0.4$ to avoid a compression failure in flexure.

If required compression steel can be used, and in this case the corresponding equation for doubly reinforced beams should be employed.

The transverse steel is now designed to resist the torsion. From equation (7.3)

$$r_o = \frac{T'}{4M_u}\ ,$$ and from the definition of r,

$$\frac{A_w}{s} = r_o \frac{A_{L1}f_{L1}}{(b - 2a_3)f_w b}$$

but

$$(b - 2a_3) \doteqdot 0.8b$$

Therefore $$\frac{A_w}{s} = \frac{T'A_{L1}}{3.2M_u b} = \frac{f_{L1}}{f_w} \qquad \ldots \qquad (7.5)$$

Top steel must be provided. It may be required to resist a mode 3 failure and if not a certain amount must, in any case, be provided to support the hoops. From the expressions for the interaction curves it can be shown that T_3 will be equal to T_1 if

$$R = \frac{2.T'}{M_u} - 1.$$

It is suggested that the minimum amount of top longitudinal steel should be given by,

$$R = 0.1\ .$$

Hence,

$$A_{L3} = (\frac{2T'}{M_u} - 1)\ A_{L1} \quad \frac{f_{L1}}{f_{L3}} \qquad \ldots\ldots\ldots(7.6)$$

To prevent a shear and torsion failure it is necessary to design the reinforcement in accordance with the A.C.I. code to resist a shear of V_{eff} , where

$$V_{eff} = V + \frac{1.6T}{b} .$$

Thus in the design of a web reinforced beam it is necessary to proportion the beam to resist a moment of M_u and a shear of V_{eff} with the additional provisos that top longitudinal steel may be necessary (see 7.6) and the web steel should not be less than the amount given by equation (7.5).

The corresponding procedure for beams with only longitudinal steel (see Chapter 4) is to design for an effective shear V_{eff} and for the actual moment M.

7.2 DESIGN PROCEDURE

The design of a beam either with or without web reinforcement and loaded in torsion combined with bending and shear may be carried out by the following procedure.

(i) The beam must be proportioned to resist an effective shear V_{eff} where,

$$V_{eff} = V + \frac{1.6T}{b}$$

(ii) If web reinforcement is not required to satisfy rule (i) it is only necessary to provide sufficient flexural capacity to resist the desired ultimate flexural moment. If web steel is used the beam is to be designed for a moment M_u, where

$$M_u = M + T'$$

$$T' = T \sqrt{1 + 2\alpha}$$

(iii) If web reinforcement is provided in accordance with rule (i), then

$$\frac{A_w}{s} = \frac{T'}{3.2 M_u b} \quad A_{L1} \quad \frac{f_{L1}}{f_w}$$

and $A_{L3} = (\frac{2T'}{M_u} - 1) \quad A_{L1}$

$$= \not< 0.1 \; A_{L1}$$

The use of these rules will be illustrated in a design example.

The required ultimate capacity is,

2000 kip. in. flexure
700 kip. in. torsion
24 kips shear

For the purpose of this example the following choices have been made:

f'_c = 3000 lb./sq. in.

$f_{L1}=f_{L3}=f_w$ = 40,000 lb./sq. in.

a_1 = 2.0

(a) Design Without Web Reinforcement.

Assume b = 20"

Rule 1.

$$V_{eff} = V + \frac{1.6T}{b}$$

$$= 24 + \frac{1.6 \times 700}{20}$$

$$= 80 \text{ kips}$$

$$V_o = 2\ bd \sqrt{f'_c}$$

$$d = \frac{80}{2x20x3000}$$

$$= 36.4$$

Adopt $h = 40"$

Rule 2.

Flexural design for

moment = 2000 kip. in.

$$A_{L1} = 1.34$$

$$= 3 \times {}^{3}/4" \text{ Ø bars}$$

Design 20" x 40" section with 3 x $^{3}/_{4}$" Ø bars.

(b) Design With Web Reinforcement

Assume b = 16"

Rule 1.

$$V_{eff} = V + \frac{1.6T}{b}$$

$$= 24 + \frac{1.6 \times 700}{16}$$

$$= 94 \text{ kips}$$

Try 16" x 24" section.

$$V_o = 2 \times 16 \times 22 \times 3000$$

$$= 38.6 \text{ kips}$$

$$V_s = 94 - 38.6 = 55.4 = \frac{2A_w f_w d}{s}$$

$$\frac{A_w}{s} = 0.03.6$$

Rule 2.

$$M_u = M + T'$$

$$T' = T \sqrt{1 + 2\alpha}$$

$$= 700 \quad 1 + \frac{2h}{b}$$

$$= 1400 \text{ kip. in.}$$

$$M_u = 2000 + 1400$$

$$= 3400 \text{ kip. in.}$$

From flexural design

$$A_{L1} = 4.27 \text{ in.}$$

$= 3 \times 1"$ Ø bars. and
$2 \times 1\ ^1/_8"$ Ø bars.

Rule 3.

$$\frac{A_w}{s} = \frac{T'}{3.2 M_u b} \quad A_{L1} \frac{f_{L1}}{f_w}$$

$$= \frac{1400}{3.2 x 3400 x 16} \quad x \quad 4.27 \ x \ \frac{40}{40}$$

$$= 0.0343.$$

This criterion is critical.

Web Steel

$$A_{L3} = \left(\frac{2T'}{M_u} - 1\right) A_{L1}$$

$$= \left(\frac{2 \times 1400 - 1}{3400}\right) 4.27$$

$$= 0$$

Nominal top steel only required hence 2 x ½" ∅

<u>Design</u> 16" x 24"

Top steel 2 x ½" ∅

Bottom steel 3 x 1" ∅ and 2 x 1½" ∅ bars.

Web Reinforcement ½" at 5½c.c.

CHAPTER 8

DEFORMATIONS IN COMBINED BENDING AND TORSION

Apart from a few qualitative statements there is little to guide the designer when calculating deflections or rotations caused by combined bending and torsion. In this chapter an attempt is made to derive equations for both deflections and rotations.

This work was based on only twelve test beams, which were designed in accordance with the rules set out in Chapter 7. However the tests cover quite a wide range of possible practical designs that can be produced by the proposed design method. In particular the range of values of $(A_{L1}-A_{L3})f_{L1}/f'_c bd$ was 0.13 to 0.37 and the ratio of torque to moment varied from 0.18 to 0.39.

At this stage only short term loads can be considered. The method is further restricted to the working load range, say up to two thirds of the ultimate load.

8.1 TORSIONAL DEFORMATIONS

The rotation of a beam loaded in combined torsion and flexure may be divided into two parts, the rotation arising from displacements across cracks and rotation caused by stresses in the intact concrete between cracks. In the analysis of forces acting in a section of a beam sustaining torsion and bending at service loads a difficulty is encountered because the torque is distributed between the concrete (T_c) and the steel acting in tension (T_s).

i.e. $T = T_c + T_s$ .... (8.1)

An attempt will firstly be made to estimate the component T_c of the torsional loading.

Prior to cracking the effect of the reinforcement on the behaviour of the beam in torsion is negligible. If desired some allowance could be made for the increased shear modulus of rigidity of steel compared with concrete, but in this treatment this effect will be ignored. Hence for torques below the cracking torque of the section,

$$T_c = T$$

and $$T_s = 0$$

It was shown in Chapter 4 that the cracking load of a plain concrete section could be computed from a maximum principal tensile stress criterion of failure. This criterion for the cracking load of reinforced beams in combined bending and torsion, has been employed in the following analysis.

The limiting tensile stress is,

$$f_t = \frac{f_b}{2} + \sqrt{\left(\frac{f_b}{2}\right)^2 + \tau^2}$$

Where f_b is the direct stress in bending and τ is the shear stress. If the expressions for the stresses are substituted in the above formula the more convenient equation below results.

$$\left(\frac{T_{cr}}{T_o}\right)^2 + \frac{M_{cr}}{M_o} = 1$$

Where T_{cr} and M_{cr} are the desired cracking loads and T_o and M_o are the cracking loads of plain concrete specimens in pure torsion and pure moment respectively.

If the above quadratic is solved for T_{cr}, then

$$\frac{T_{cr}}{T_o} = -\frac{T_o}{2\psi M_o} + \sqrt{\left(\frac{T_o}{2\psi M_o}\right)^2 + 1} \qquad \ldots \quad (8.2)$$

$$M_{cr} = \frac{T_{cr}}{\psi}$$

After cracking the system of forces that are sustaining the torque (T_c) independently of steel tensile stresses are shown in Figure 8.1. In fact these forces are the same as those which resist the applied torque in beams without web reinforcement. It would therefore seem reasonable to take the ultimate capacity of a beam containing only longitudinal reinforcement

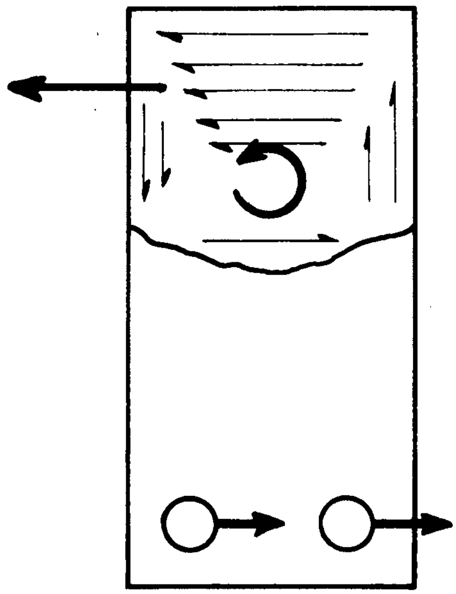


FIG. 8.1. FORCES IN SECTION RESISTING T_c AFTER CRACKING.

as the upper limit to the torque T_c that can be resisted by the mechanism in Figure 8.1. Furthermore it was found in Chapter 4 that the torsional capacity of an equivalent plain concrete specimen was a good estimate of the failure torque of a beam containing only longitudinal steel loaded in bending and torsion. It has been somewhat arbitarily assumed that the maximum resistance of T_c would be mobilised at an applied torque of 90% of the ultimate capacity of the section.

Thus $\quad T_c = T_o$

when $\quad T = 0.9\ T_u$

where T_u is the ultimate torsional capacity of the section.

For torques intermediate between T_{cr} and 0.9 T_u it has been assumed that the component of the torque resisted by the concrete would vary linearly.

It is not of course possible to measure the actual values of T_c in a test beam. However some verification of the above method is possible if the web steel strain results are used.

Various formulae have been proposed expressing the loads in terms of longitudinal and transverse steel stresses. The relationship between the torque and the web steel stress for the theories of Cowan, Rausch, Andersen, Gesund and Yudin may be expressed as,

$$f_w = \frac{s}{A_w} \frac{T_s}{\lambda b'd'} \qquad \ldots \qquad (8.3)$$

In this section the widely used method of Rausch will be employed.

$$f_w = \frac{s}{A_w} \frac{T_s}{2b'd'} \qquad \ldots \qquad (8.4)$$

or in terms of strain,

$$T_s = \frac{A_w}{s} \cdot 2b'd'e_w E_w \qquad \ldots \qquad (8.5)$$

As this relationship may be taken as fairly reliable, it can be employed to compute the values of T_s from the measured steel strains. The value of T_c may then be deduced from the values of T_s and the applied torque.

i.e. $T_c = T - T_s$

In Figure 8.2, T_c is compared with the total torque (expressed as a fraction of the failure torque). It can be seen from this figure that the above method represents a reasonable approximation to the actual variation of T_c.

As T_s is negligible prior to cracking, the torsional stiffness in this range may be computed from the elastic theory. After cracking expressions relating the stiffness to the web steel strains must be employed.

(a) Elastic

The rotation per unit length of an elastic homogeneous beam is given by

$$\theta = \frac{T}{GJ}$$

J is based simply on the geometry of the section, and may be calculated from the relationship

$$J = K_2 \, b^3 \, h$$

h/b	K_2
1.0	.141
1.2	.166
1.4	.187
1.6	.204
1.8	.217
2.0	.229
2.5	.249
3.0	.264

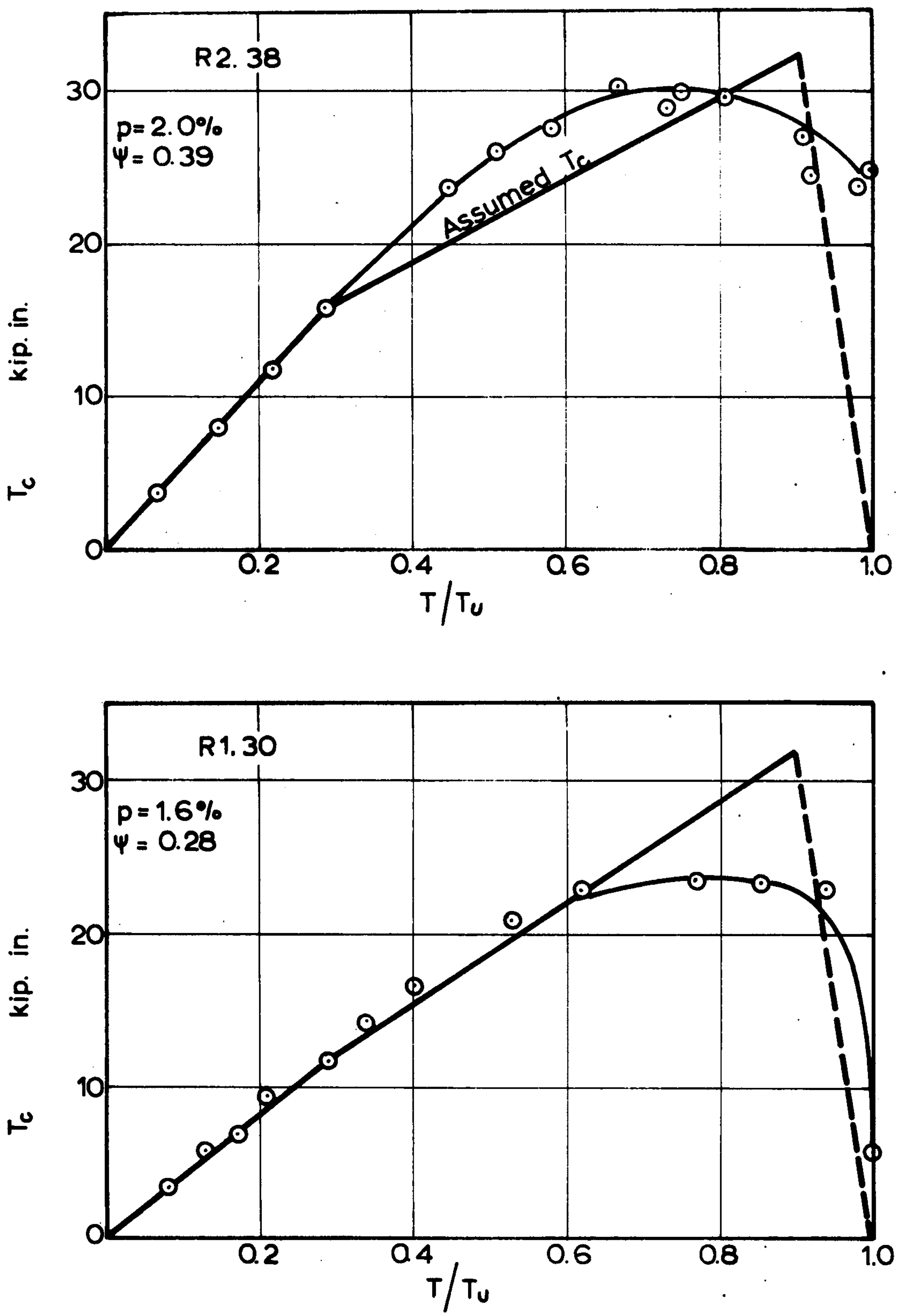


FIG. 8.2 VARIATION OF T_c WITH T/T_u

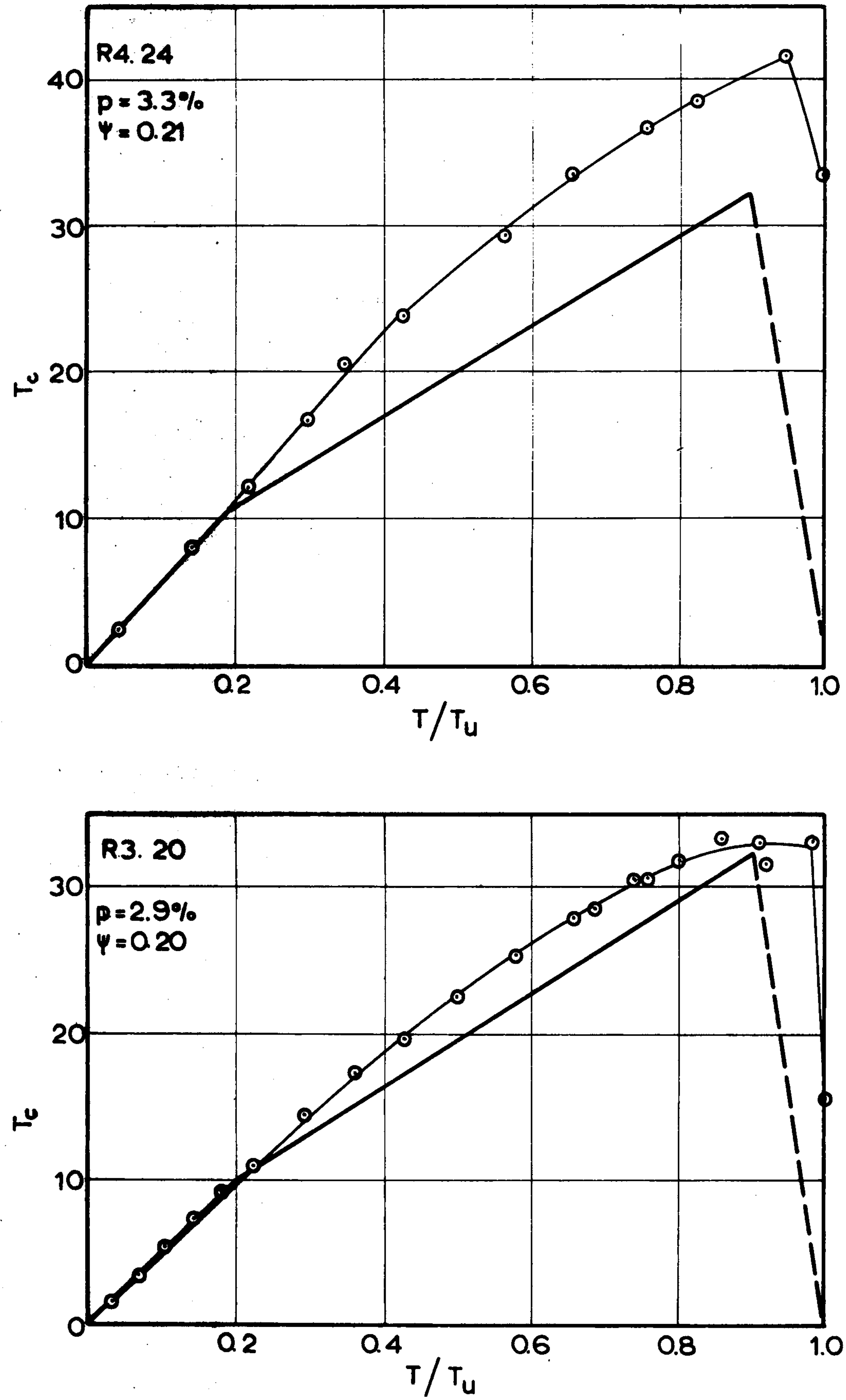
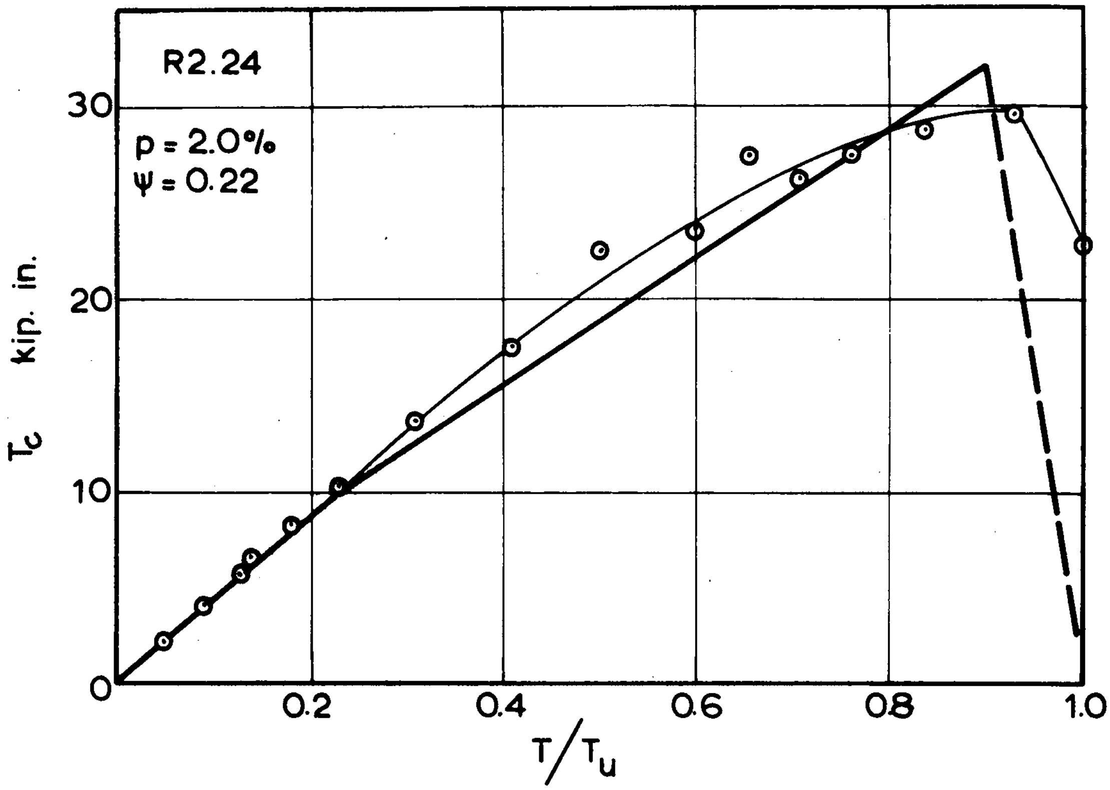


FIG. 8.2 CONTINUED



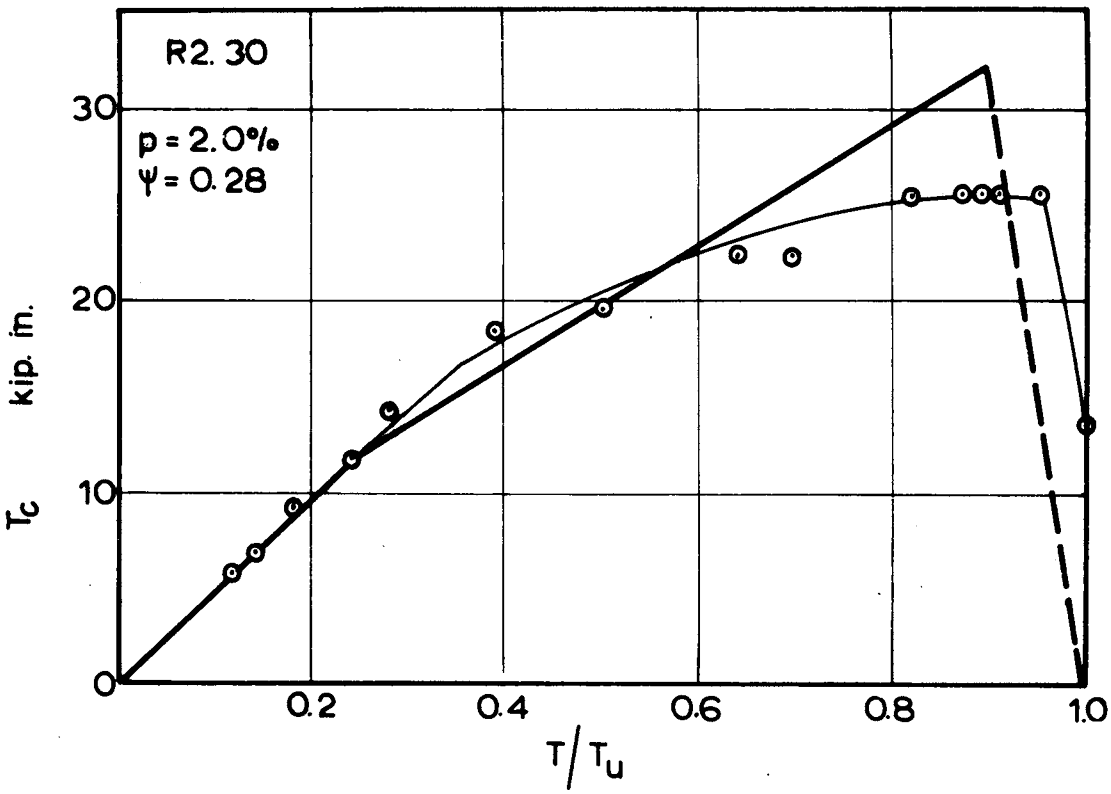
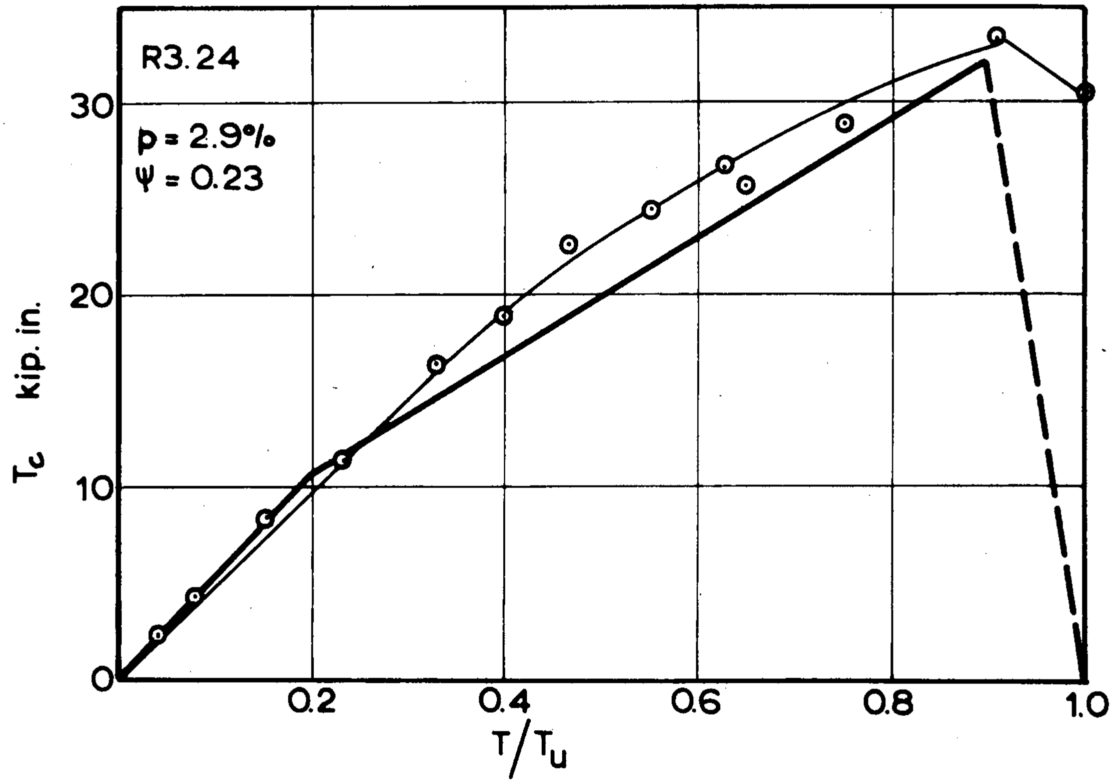


FIG. 8.2 CONTINUED



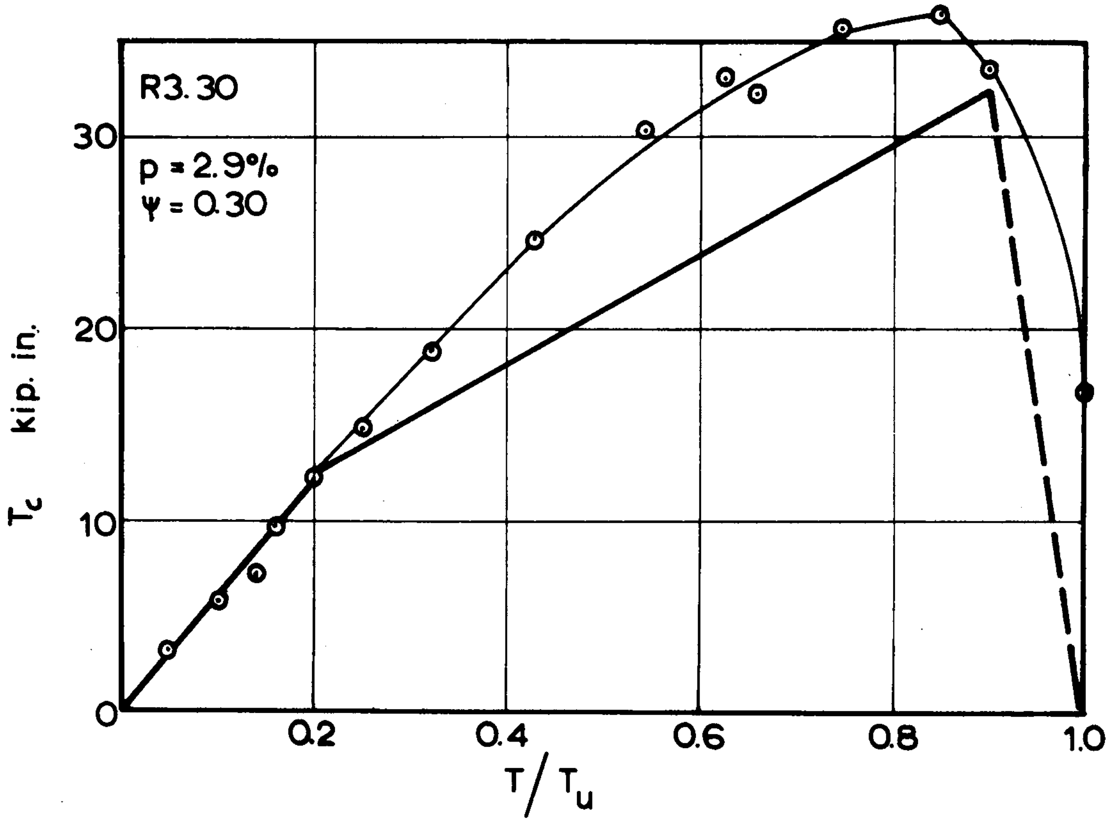


FIG. 8.2 CONTINUED

The shear modulus of elasticity (G) is a property of the material. For design purposes its value may be taken as,

$$G = \frac{E_c}{2}$$

(b) Cracked Stiffness

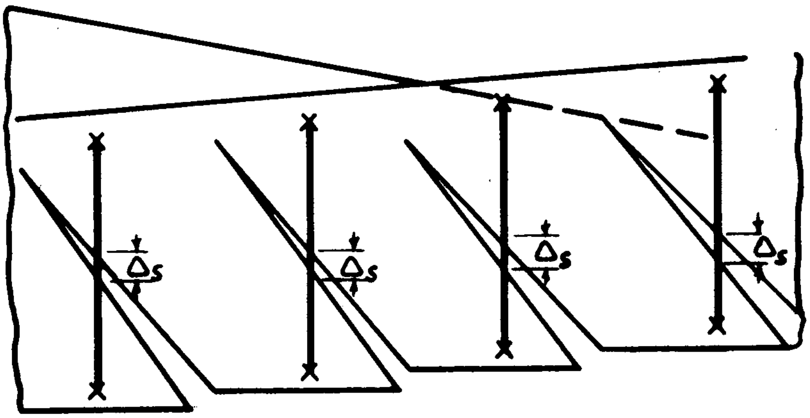
After cracking the stiffness is reduced. In Figure 8.3 a side elevation of the beam is given showing the displacements associated with cracking. If a parabolic distribution of stress, as suggested by Cowan (reference 8.1), is assumed then the vertical displacement which occurs at each stirrup is,

$$\Delta_s = \frac{2}{3} e_w d'$$

The vertical displacement per unit length of beam may be written as,

$$\Delta = \frac{\Delta_s}{s}$$

$$= \frac{2}{3} \frac{e_w d'}{s}$$



Stirrups

SIDE ELEVATION

FIG. 8.3 ROTATION PRODUCING STEEL DEFORMATIONS

As the displacement on the other side is equal but opposite in sense, the rotation per unit length (θ_s) is,

$$\theta_s = \frac{\Delta}{\frac{b'}{2}}$$

$$= \frac{4}{3}\frac{e_w d'}{s\ b'} \qquad \ldots \qquad (8.6)$$

In addition to rotation due to displacements across the cracks, rotation is possible due to deformations in the intact concrete between cracks. An approximate estimate of this rotation is,

$$\theta_c = \frac{T_c}{GJ} \qquad \ldots \qquad (8.7)$$

The total rotation in the cracked state is then,

$$\theta = \theta_c + \theta_s \qquad \ldots \qquad (8.8)$$

It is of interest to evaluate this method before the errors involved in calculating the steel stresses are included. Figure 8.4 has been prepared to compare the measured torque rotation results with the theoretical predictions. In preparing this figure the values of the web steel strains required in equation 8.6 were obtained from actual strain gauge readings. The correlation between the experimental results and the theoretical predictions based on this method is excellent.

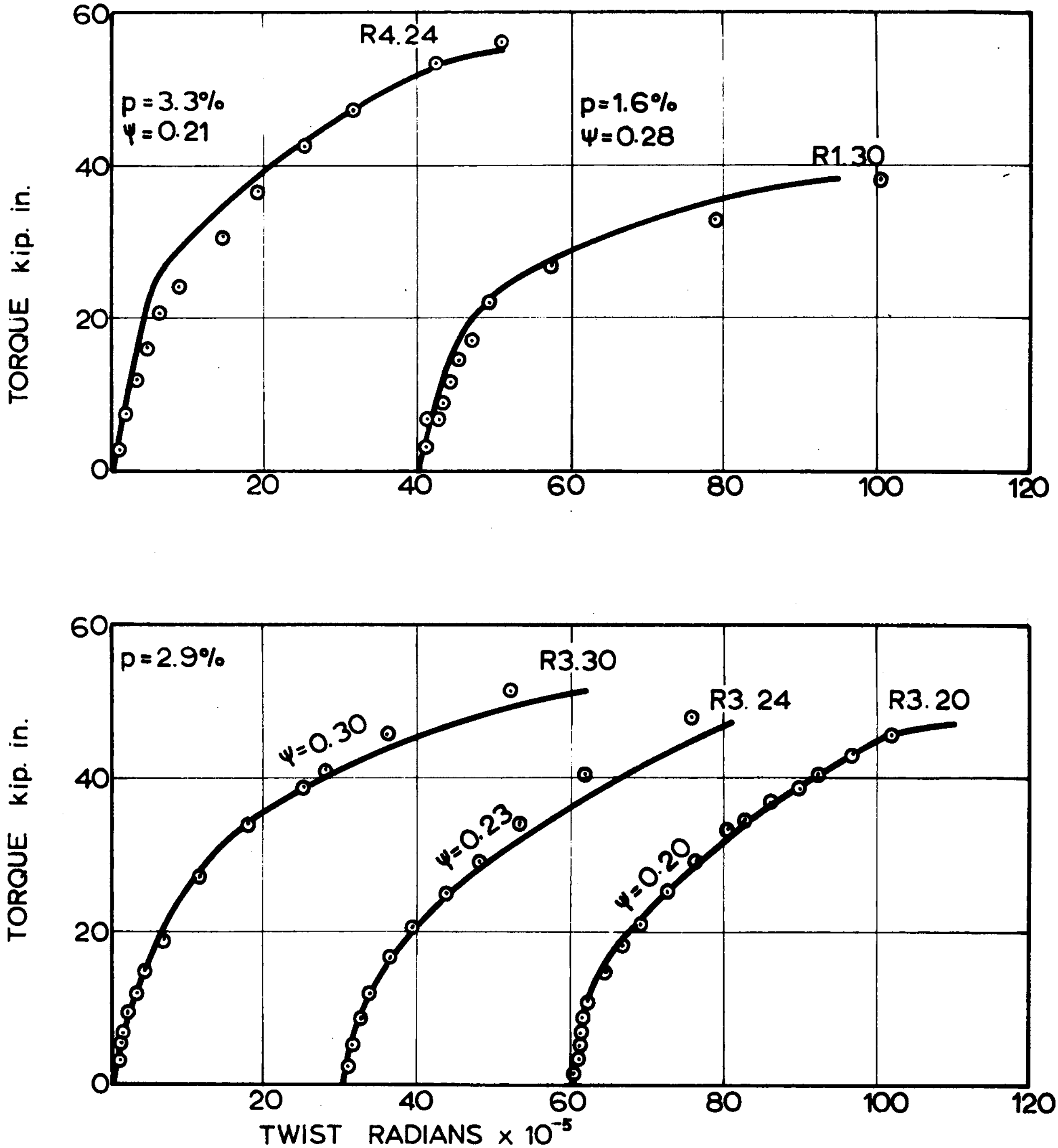


FIG. 8.4. EXPERIMENTAL RESULTS — TWIST COMPUTED FROM MEASURED STEEL STRAINS.

(cont. over)

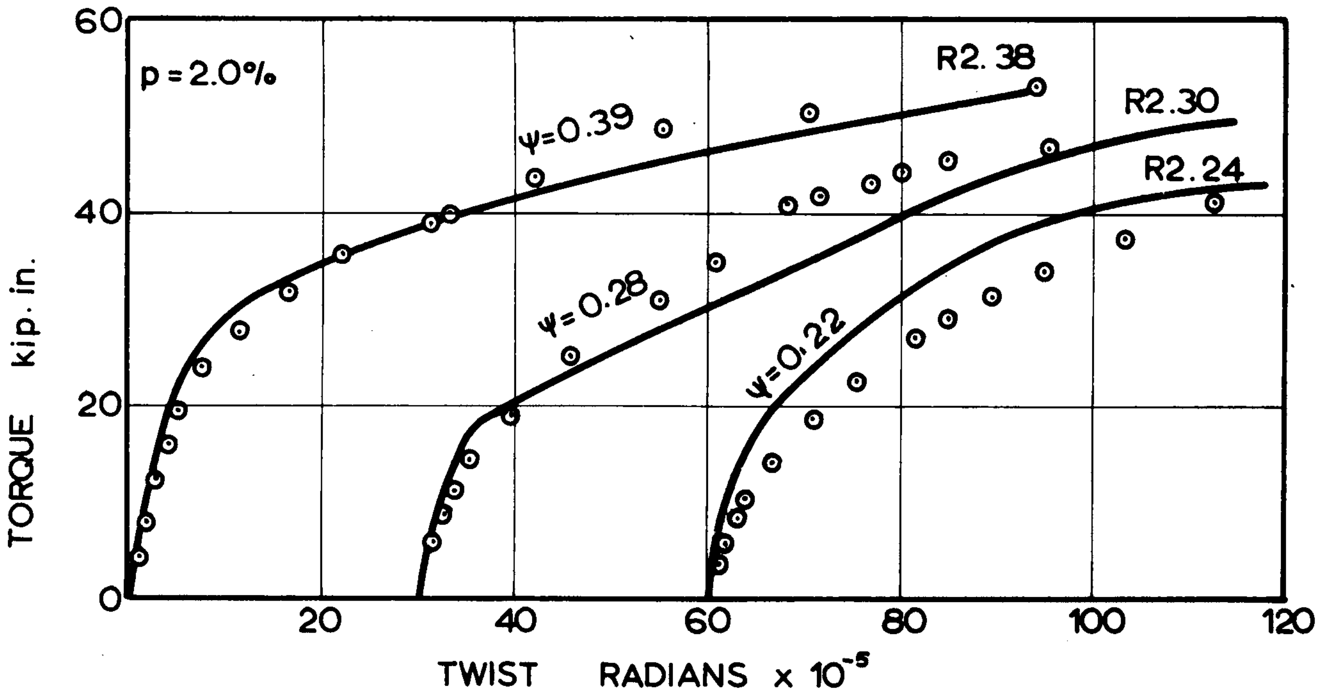


FIG. 8.4 CONTINUED

It can therefore be concluded that if an accurate method for predicting steel strains was available a good estimate of the rotation could be made. Many formulae similar to equation 8.3 have been proposed. Any of these would probably give a sufficiently accurate relationship between the steel stress and T_s the torque resisted by the steel. Unfortunately the far more significant problem of determining what proportion of the applied torsional loading is resisted by the steel, has been ignored.

If the empirical estimate of T_c and hence T_s given earlier is used, the following method can be derived.

When equation 8.6 is expressed in terms of stress,

$$\theta_s = \frac{4\ f_w d'}{3\ sb'E_w} \qquad \ldots \qquad (8.9)$$

Now if the equation for f_w (eq 8.4) is substituted in equation 8.9, then,

$$\theta_s = \frac{4\ d'}{3\ sb'E_w} \cdot \frac{s\ T_s}{A_w . 2b'd'}$$

$$\theta_s = \frac{2}{3} \cdot \frac{T_s}{A_w E_w (b')^2} \qquad \ldots \qquad (8.10)$$

Thus from equations 8.7 and 8.8,

$$\theta = \frac{2}{3} \cdot \frac{T_s}{A_w E_w (b')^2} + \frac{T_c}{GJ} \qquad \ldots \qquad (8.11)$$

The accuracy of this formula is illustrated in Figure 8.5. In this case due to innaccuracies in T_s resulting from innaccuracies in the empirical estimate of T_c correlation between the experimental and theoretical results is only fair. It should be appreciated that beams R4.20 and R4.24 were extremely heavily reinforced and are not likely to occur commonly in practice. For the more lightly reinforced beams particularly at moderate ratios of torsion to bending the proposed method provides quite good estimates of the torsional rotations.

8.2 FLEXURAL DEFLECTIONS

In Chater 2.4 some of the main features of the theories for deflection were discussed. It was found in that chapter that a good empirical approximation to the effective moment of inertia was given by Branson as,

$$I_{eff} = \left(\frac{M_{cr}}{M}\right)^4 I_{gt} + \left(1 - \left(\frac{M_{cr}}{M}\right)^4\right) I_{cr} \not> I_{gt}$$

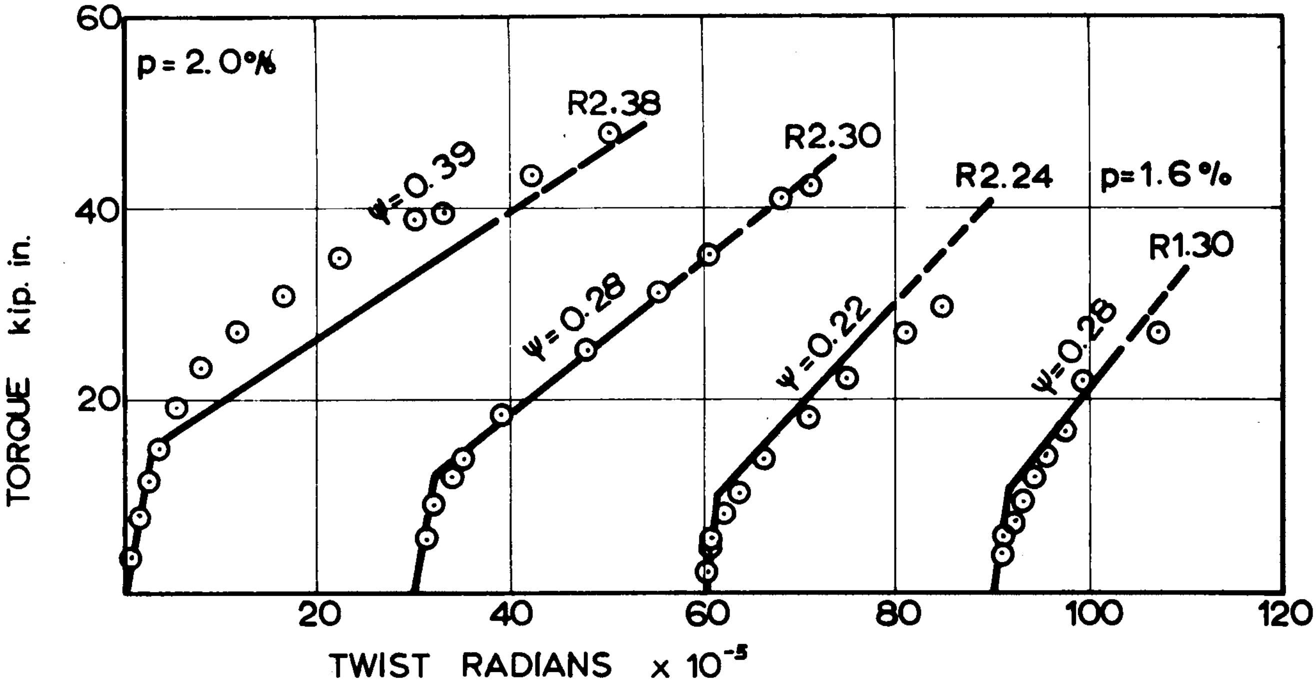


FIG. 8.5 COMPARISON OF EXPERIMENTAL RESULTS AND THEORETICAL TORQUE TWIST CURVES.

(Continued over Page)

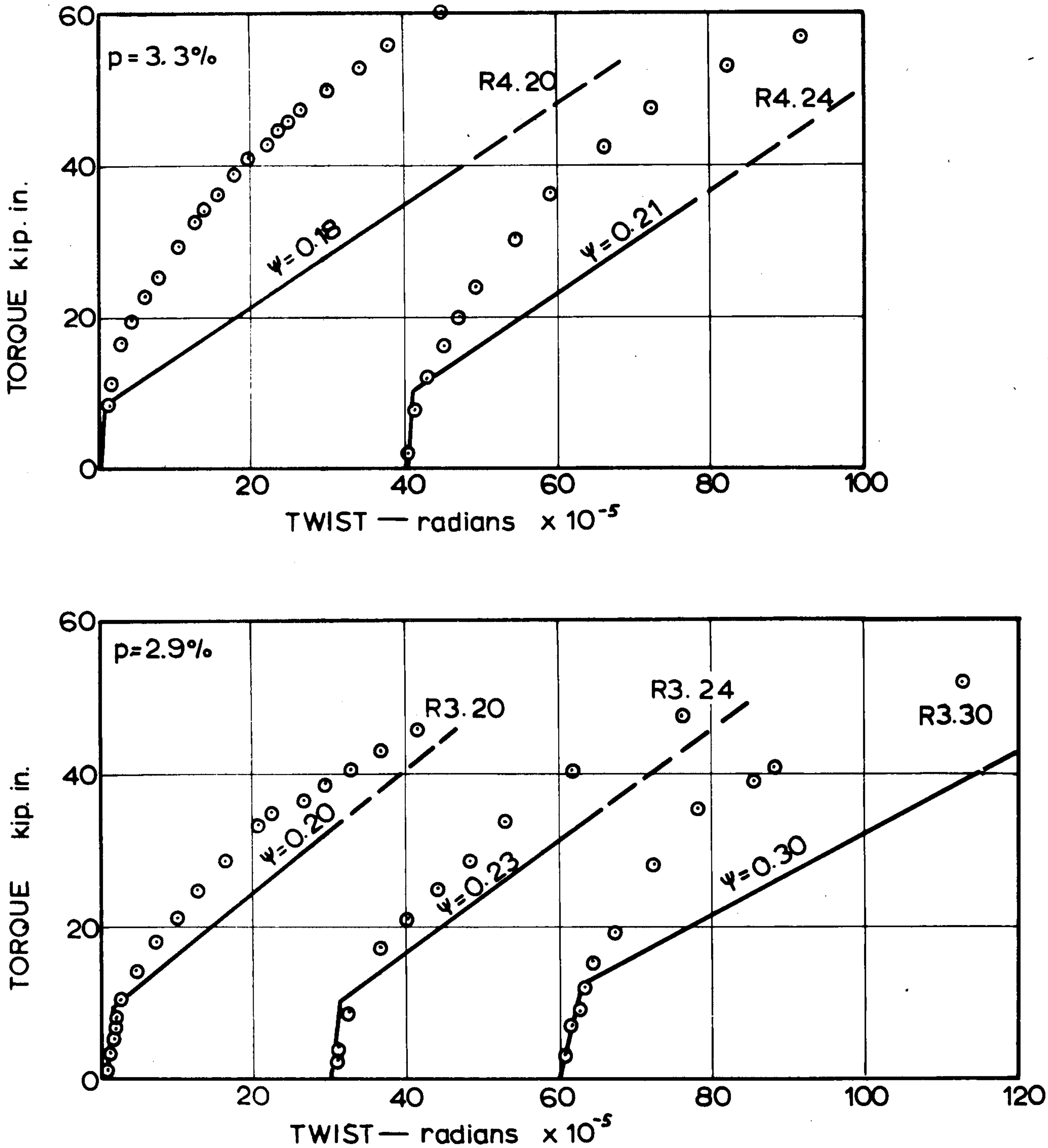


FIG. 8.5 CONTINUED

Branson further found that instead of using a varying I_{eff} along the beam a constant I'_{eff} which allows for the stiffer uncracked sections, may be used.

$$I'_{eff} = \left(\frac{M_{cr}}{M}\right)^3 I_{gt} + \left(1 - \left(\frac{M_{cr}}{M}\right)^3\right) I_{cr} > I_{gt} \qquad \ldots \qquad (8.12)$$

This equation is suitable for the case of bending and torsion if the values of I_{cr} and M_{cr} are modified. The cracking moment in bending and torsion can calculated from equation 8.2. An expression for the cracked stiffness can be derived if consideration is given to the longitudinal steel stresses in the combined loading case. It might be recalled that in the derivation of the formula for the torsional deformations the theory of Rausch was employed. The corresponding stress in the longitudinal steel may be deduced from the equations for the area of longitudinal steel.

$$A_{LT} = \frac{(b' + d')\ T_s}{2f_{L1}b'd'} \qquad \ldots \qquad (8.13)$$

$$A_{LB} = \frac{M}{f_{L1}jd} \qquad \ldots \qquad (8.14)$$

Where A_{LT} is the contribution to the main longitudinal steel due to torsion and A_{LB} is the contribution due to bending.

The total area of main longitudinal steel, A_{L1}, is then,

$$A_{L1} = A_{LT} + A_{LB}$$

$$= \frac{(b' + d')T_s}{2f_{L1}b'd'} + \frac{M}{f_{L1}jd}$$

If this expression is rearranged we obtain,

$$f_{L1} = \frac{(b' + d')\ T_s}{2A_{L1}b'd'} + \frac{M}{A_{L1}jd}$$

$$= \frac{M}{A_{L1}jd}\left(1 + \frac{jd(b' + d')}{2b'd'} \cdot \frac{T_s}{M}\right) \qquad \ldots \qquad (8.15)$$

Although the exact value of T_s in this formula is not known, an examination of the equation suggests that the stress and hence the strain in the longitudinal steel would be increased by the presence of torsion. In fact in Chapter 3, Figure 3.32, it was found that the strains in the longitudinal steel would be increased by the presence of torsion.

From the geometry of the flexural deformation, see Figure 8.6,

$$\phi_{cr} = \frac{e_s}{d(1 - k)}$$

This can be expressed in the customary form by putting,

$$\phi_{cr} = \frac{M}{EI_{cr}}$$

Then

$$EI_{cr} = \frac{d(1 - k)M}{e_s}$$

Now

$$e_s = \frac{f_{L1}}{E_s}$$

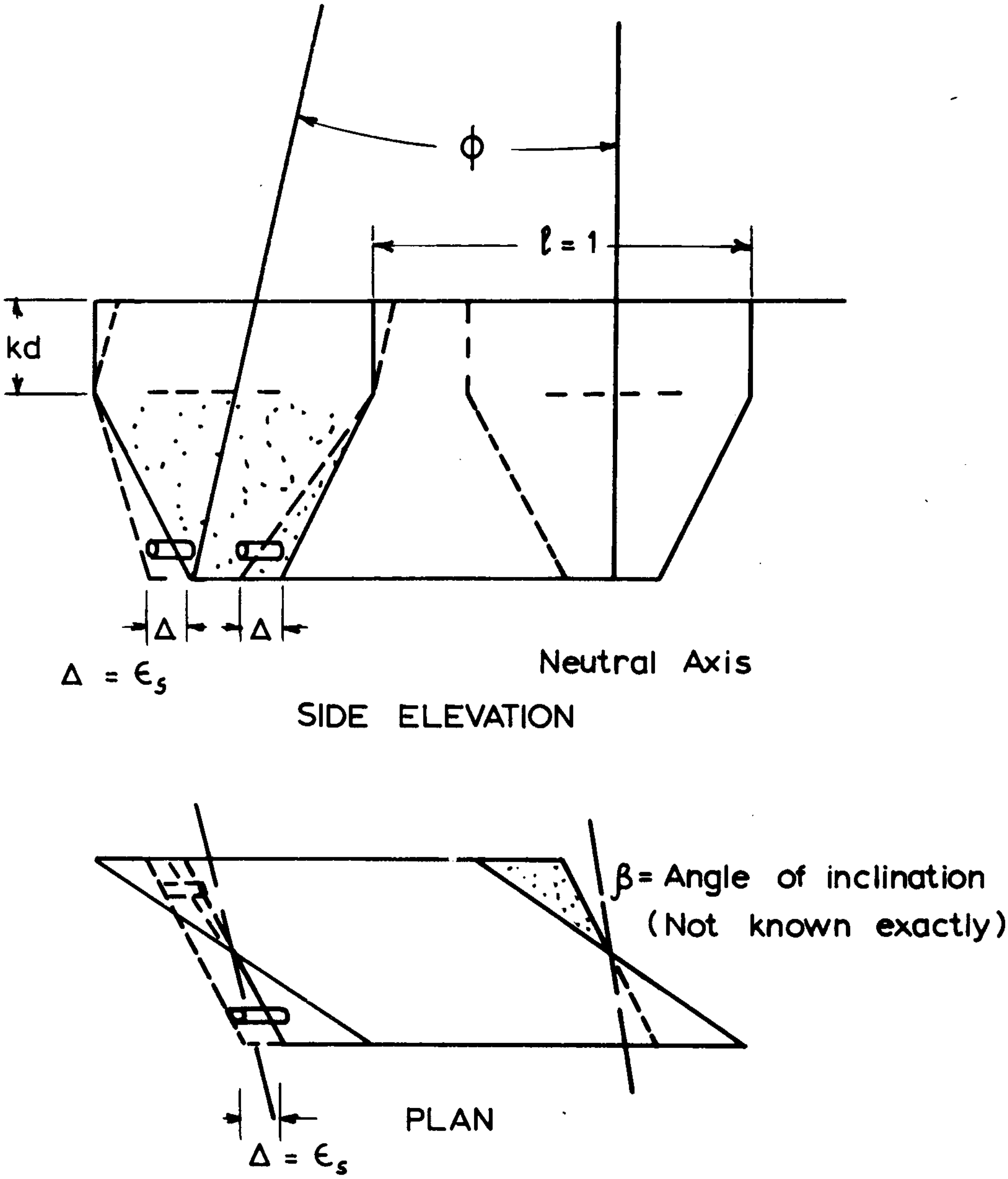


FIG. 8.6. GEOMETRY OF BENDING DEFORMATIONS

Therefore

$$EI_{cr} = \frac{d(1 - k)E_s M}{f_{L1}} \qquad \ldots \qquad (8.16)$$

When the value of f_{L1} (equation 8.15) is substituted into equation 8.16, then

$$EI_{cr} = \frac{A_{L1} jd(1 - k)dE_s}{1 + \frac{jd(b' + d')}{2b'd'} \cdot \frac{T_s}{M}} \qquad \ldots \qquad (8.17)$$

It is of interest to compare this expression with the corresponding result for pure flexure.

$$EI_{cr} = A_{L1} jd(1 - k)dE_s \qquad \ldots \qquad (8.18)$$

It can be seen that the equation for bending and torsion for the limiting case of $T_s/M=0$ reduces to the equation for pure flexure. For other values of T_s/M the above method predicts that torsion will increase the deflections of a beam.

Using the values of T_s given by the empirical method discussed in the previous section and Branson's method a comparison between experimental deflections and the deflections predicted by equation 8.17 can be made. This comparison is shown in Figure 8.7 where it can be seen that the agreement between test results and the theory is very good.

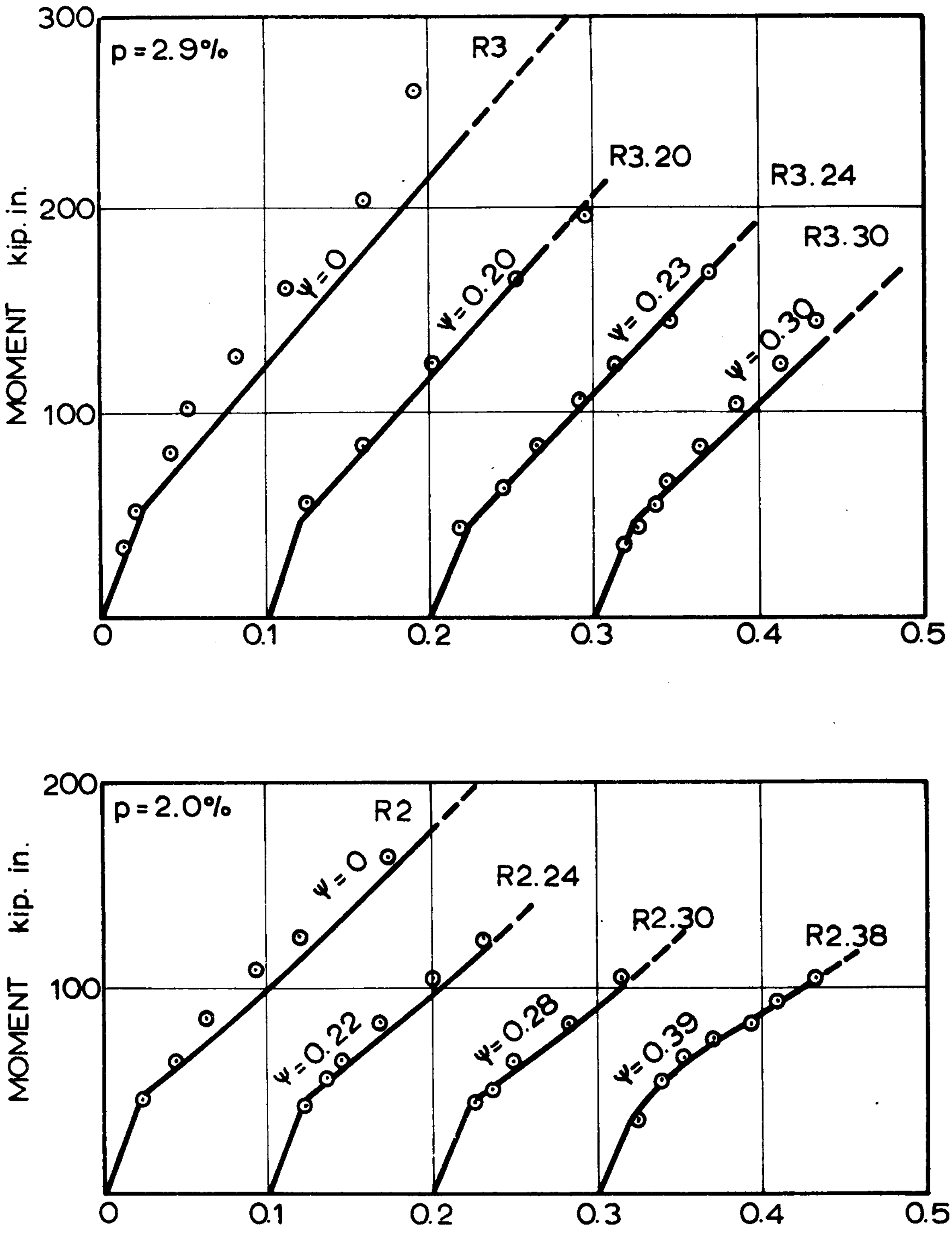


FIG. 8.7 EXPERIMENTAL RESULTS — PREDICTED LOAD DEFLECTION CURVES.

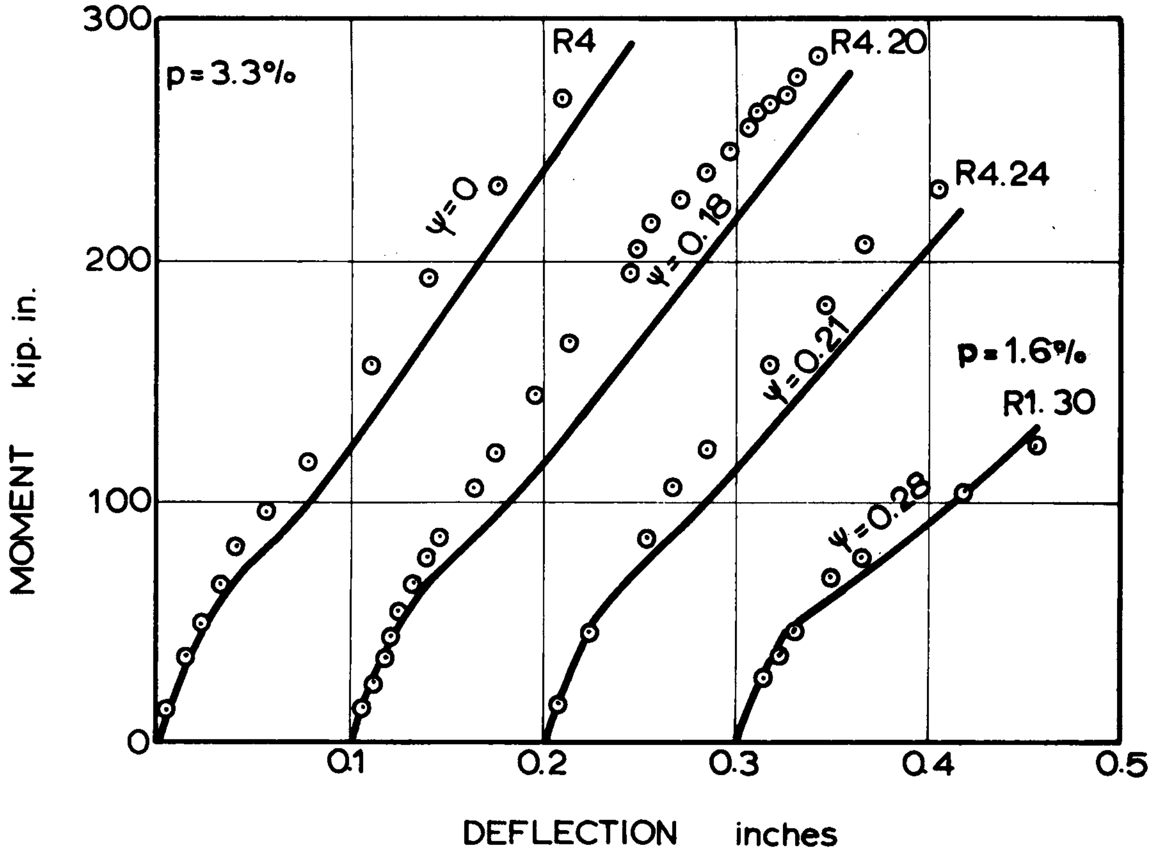


FIG. 8.7 CONTINUED

CHAPTER 9

CONCLUSIONS

A solution to the problem of "The Strength and Stiffness of Rectangular Reinforced Concrete Beams Subjected to Combined Flexure and Torsion." has been presented. This solution has been obtained from an extensive experimental investigation, a rational analysis of failure mechanisms, empirical treatment of test data and where necessary engineering assumptions.

In particular the following conclusions were reached:

1. Beams containing only longitudinal steel.

1.1 The ultimate torsional strength of a beam without web reinforcement was found to be given by a plastic stress distribution and a maximum stress criterion of failure.

1.2 Based on an analysis of all the available test results for plain concrete beams loaded in pure torsions, a safe and reasonably accurate expression for the critical tensile strength of concrete in torsion has been presented.

1.3 For beams reinforced in the longitudinal direction only, it was shown to be adequate to assume no interaction between bending and torsion and a linear interaction between shear and torsion.

1.4 A design method which replaces the torsional loading on a beam with only longitudinal reinforcement by an equivalent effective shear force has been developed.

1.5 The theory recommended for beams without web reinforcement has been verified by comparison with the large number of tests of this investigation and with the available test results reported in the literature.

2. Ultimate strength of beams containing both longitudinal and transverse reinforcement.

2.1 A method of analysis, based on observed failure mechanisms, has been derived for the ultimate strength of web reinforced beams subjected to combined torsion, bending and moderate shear force.

2.2 An empirical equation has been adopted for the case of combined shear and secondary torsion.

2.3 The proposed analysis equations have been shown to explain satisfactorily the bending torsion interaction of web reinforced beams.

2.4 A simple ultimate strength design method has been developed from the analysis equations.

2.5 Good correlation has been found between a large number of test results and the proposed theory.

3. Deformations of web reinforced beams in combined bending and torsion.

3.1 A quite accurate theory for the flexural deflections in the presence of torsion, at service loads, has been proposed.

3.2 A reasonably satisfactory solution to the problem of rotations of web reinforced members loaded in bending and torsion has been presented.

Finally it should be emphasised that the ultimate strength design methods that have been proposed are simple, reliable and accurate.

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6.2 (see reference 2.16)

APPENDIX B

This appendix contains a summary of all the experimental data used in the comparisons of theory and experiment in Chapters 2 and 6 for beams with web reinforcement. Further details of these test results can be found in the references cited in Appendix A. A discussion of the test results of this investigation is, of course, given in Chapter 3. In the table the concrete strengths have been expressed in terms of the cylinder compressive strength. Where the investigator specified the concrete used by its cube strength a conversion factor of 0.8 has been employed.

APPENDIX B. EXPERIMENTAL DATA

		Geometry (inches)				Web Steel			Longitudinal Steel				Failure Loads		
Invest-igator	Beam No.	h	b	a_1	a_2	A_w sq. in.	f_w k.s.i.	s in.	A_{L1} sq. in.	f_{L1} k.s.i	R	f'_c p.s.i.	T kip. in.	M kip. in.	V kips
This Investigation	RE1	10.0	6.5	1.6	1.7	0.110	49.0	3.00	0.392	44.0	1.000	4599	81.4	6.3	0.18
	RE2	10.0	6.5	1.6	1.3	0.110	49.0	3.00	0.392	44.0	1.000	4599	83.5	32.0	0.92
	RE3	10.0	6.5	1.6	1.3	0.110	49.0	3.00	0.392	44.0	1.000	4599	81.5	45.0	1.28
	RE4	10.0	6.5	1.6	1.3	0.110	49.0	3.00	0.392	44.0	1.000	4599	74.6	84.4	3.26
	RE5	10.0	6.5	1.6	1.3	0.110	49.0	3.00	0.392	44.0	1.000	4599	66.0	108.2	3.07
	RE4*	10.0	6.5	1.6	1.3	0.110	49.0	3.00	0.392	44.0	1.000	4599	38.0	134.0	3.86
	RU1	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	3679	73.3	6.3	0.18
	RU3A*	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	4629	76.0	6.3	0.19
	RU2	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	3679	84.9	51.1	0.0
	RU3	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	3679	105.0	84.0	0.0
	RU3A	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	4629	89.4	149.3	0.0
	RU4	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	3679	85.5	145.0	4.13
	RU5	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	3679	75.4	249.7	7.15
	RU5A	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	4399	68.3	266.8	7.80
	RU6	10.0	6.5	1.8	1.4	0.110	49.0	4.00	0.880	46.8	0.239	3679	59.1	281.2	7.95
	36T4	10.0	6.5	1.6	1.3	0.110	43.0	4.00	0.880	37.7	0.295	4399	62.6	240.4	7.53
	36T4C	10.0	6.5	1.6	1.4	0.110	43.0	4.00	0.880	37.7	0.309	4339	94.1	61.1	0.0
	36T5.5	10.0	6.5	1.6	1.4	0.110	43.0	5.50	0.880	37.7	0.295	4629	85.9	173.4	0.0
	77T5	10.0	6.5	1.7	1.4	0.110	43.0	5.00	1.200	37.7	1.000	4629	91.6	262.4	0.0
	77T4	10.0	6.5	1.7	1.4	0.110	43.0	4.00	1.200	37.7	1.000	4629	107.6	223.4	0.0
	24T3	10.0	6.5	1.3	1.4	0.110	43.0	3.00	0.392	47.1	0.372	4339	70.8	46.6	0.0
	38T5	10.0	6.5	1.7	1.6	0.110	43.0	5.00	1.570	38.7	0.156	3829	80.1	216.4	6.73

APPENDIX B. EXPERIMENTAL DATA

		Geometry (inches)				Web Steel			Longitudinal Steel				Failure Loads		
Investigator	Beam No.	h	b	a_1	a_2	A_w sq. in.	f_w k.s.i.	s in.	A_{L1} sq. in.	f_{L1} k.s.i.	R	f'_c p.s.i.	T kip. in.	M kip. in.	V kips
This Investigation	R4.20	10.0	5.0	1.5	1.2	0.049	58.6	2.59	1.420	40.6	0.155	3474	59.9	331.0	0.0
	R4.24	10.0	5.0	1.5	1.2	0.049	58.6	2.25	1.420	40.6	0.155	3034	56.5	264.0	0.0
	R3.20	10.0	5.0	1.5	1.1	0.049	58.6	3.12	1.220	40.6	0.180	3444	50.7	252.0	0.0
	R3.24	10.0	5.0	1.5	1.1	0.049	58.6	2.72	1.220	40.6	0.180	3157	53.7	230.0	0.0
	R3.30	10.0	5.0	1.5	1.1	0.049	58.6	2.38	1.220	40.6	0.180	3414	61.6	207.0	0.0
	R2.24	10.0	5.0	1.7	1.2	0.049	58.6	3.80	0.830	42.4	0.265	3239	44.2	205.0	0.0
	R2.30	10.0	5.0	1.7	1.2	0.049	58.6	3.32	0.830	42.4	0.265	3314	49.7	176.0	0.0
	R2.38	10.0	5.0	1.7	1.2	0.049	58.6	2.84	0.830	42.4	0.265	3474	53.4	138.0	0.0
	R1.30	10.0	5.0	1.2	1.2	0.049	58.6	3.95	0.710	41.3	0.310	3279	41.8	146.0	0.0
	R4A	10.0	5.0	1.5	1.2	0.049	58.6	3.22	1.420	40.6	0.155	2954	50.5	92.1	3.94
	R4B	10.0	5.0	1.5	1.2	0.049	58.6	3.22	1.420	40.6	0.155	2954	51.6	166.0	6.92
	R4.20A	10.0	5.0	1.5	1.2	0.049	73.6	2.00	1.420	40.6	0.155	3474	72.0	220.0	9.08
	R4.20B	10.0	5.0	1.5	1.2	0.049	73.6	2.00	1.420	40.6	0.155	3474	70.8	151.0	6.30
	R4.24A	10.0	5.0	1.5	1.2	0.049	58.6	2.00	1.420	40.6	0.155	3034	62.1	255.0	8.21
	R3A	10.0	5.0	1.5	1.1	0.049	73.6	3.00	1.220	40.6	0.180	2919	57.6	92.1	3.94
	R3B	10.0	5.0	1.5	1.1	0.049	73.6	3.00	1.220	40.6	0.180	2919	51.5	166.0	6.92
	R3.20A	10.0	5.0	1.5	1.1	0.049	58.6	2.50	1.220	40.6	0.180	3444	61.7	183.0	7.60
	R3.20B	10.0	5.0	1.5	1.1	0.049	58.6	2.50	1.220	40.6	0.180	3444	59.0	78.9	3.41
	R3.24A	10.0	5.0	1.5	1.1	0.049	73.6	2.50	1.220	40.6	0.180	3157	56.5	92.9	3.97
	R3.24B	10.0	5.0	1.5	1.1	0.049	73.6	2.50	1.220	40.6	0.180	3157	54.9	158.0	6.57
	R3.30A	10.0	5.0	1.5	1.1	0.049	73.6	2.00	1.220	40.6	0.180	3414	62.2	203.0	8.39
	R3.30B	10.0	5.0	1.5	1.1	0.049	73.6	2.00	1.220	40.6	0.180	3414	63.4	86.8	3.73
	R2A	10.0	5.0	1.7	1.2	0.049	58.6	3.00	0.830	42.4	0.265	3199	55.7	111.0	4.71
	R2B	10.0	5.0	1.7	1.2	0.049	58.6	3.00	0.830	42.4	0.265	3199	50.9	78.7	3.42
	R2.24A	10.0	5.0	1.7	1.2	0.049	58.6	3.30	0.830	42.4	0.265	3239	62.3	140.0	5.85
	R2.30A	10.0	5.0	1.7	1.2	0.049	73.6	3.00	0.830	42.4	0.265	3314	50.3	104.0	4.43
	R2.30B	10.0	5.0	1.7	1.2	0.049	73.6	3.00	0.830	42.4	0.265	3314	50.6	71.8	3.13
	R2.38A	10.0	5.0	1.7	1.2	0.049	73.6	2.50	0.830	42.4	0.265	3474	54.6	120.0	5.08
	R2.38B	10.0	5.0	1.7	1.2	0.049	73.6	2.50	0.830	42.4	0.265	3474	48.8	70.7	3.08
	R1.30A	10.0	5.0	1.2	1.2	0.049	58.6	3.14	0.710	41.3	0.310	3279	42.6	97.1	4.13
	R1.30B	10.0	5.0	1.2	1.2	0.049	58.6	3.14	0.710	41.3	0.310	3279	40.9	61.8	2.73

APPENDIX B. EXPERIMENTAL DATA

Investigator	Beam No.	Geometry (inches) h	b	a_1	a_2	Web Steel A_w sq.in.	f_w k.s.i.	s in.	Longitudinal Steel A_{L1} sq.in.	f_{L1} k.s.i.	R	f'_c p.s.i.	Failure Loads T kip. in.	M kip. in.	V kips
Collins	V1	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	38.2	544.0	22.30
	V1*	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	60.6	421.0	17.30
	V2	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	52.3	413.0	17.00
	V2*	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	86.0	166.0	7.10
	V3	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	16.9	685.0	27.80
	V3*	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	79.2	394.0	16.20
	V4	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	91.3	243.0	10.20
	V4*	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	40.4	664.0	27.00
	V5*	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	89.7	0.0	0.0
	V6	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	24.8	668.0	27.20
	V6*	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	82.9	348.0	14.40
	V7	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	83.9	298.0	9.30
	V7*	10.0	6.5	1.5	1.3	0.049	58.6	4.75	1.840	65.2	0.333	5029	103.4	210.0	12.10
	U1	10.0	6.5	1.5	1.3	0.049	58.6	3.00	1.840	65.2	0.333	5029	106.6	0.0	0.0
	U1*	10.0	6.5	1.5	1.3	0.049	58.6	3.00	1.840	65.2	0.333	5029	99.5	199.0	8.40
	U2	10.0	6.5	1.5	1.3	0.049	58.6	3.00	1.840	65.2	0.333	5029	43.9	689.0	28.00
	U2*	10.0	6.5	1.5	1.3	0.049	58.6	3.00	1.840	65.2	0.333	5029	85.2	315.0	13.10
	U3	10.0	6.5	1.5	1.3	0.049	58.6	3.00	1.840	65.2	0.333	5029	82.7	533.0	21.80
	U3*	10.0	6.5	1.5	1.3	0.049	58.6	3.00	1.840	65.2	0.333	5029	66.2	720.0	29.30
	T1	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	75.1	327.0	13.50
	T2	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	92.9	102.0	4.10
	T4	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	53.0	523.0	21.40
	T4*	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	81.3	366.0	15.10
	T5	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	63.4	432.0	17.70
	T5*	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	75.4	0.0	0.0
	T6	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	83.0	196.0	8.30
	T6*	10.0	6.5	1.2	1.7	0.049	58.6	3.62	1.570	39.3	0.140	5029	29.4	584.0	23.90

APPENDIX B. EXPERIMENTAL DATA

Invest-igator	Beam No.	Geometry (inches)				Web Steel			Longitudinal Steel				Failure Loads		
		h	b	a_1	a_2	A_w sq. in.	f_w k.s.i.	s in.	A_{L1} sq. in.	f_{L1} k.s.i.	R	f'_c p.s.i.	T kip. in.	M kip. in.	V kips
Ernst	3TR3	12.0	6.0	1.3	1.0	0.049	55.5	14.00	0.220	53.6	1.000	3922	34.3	0.0	0.0
	3TR7	12.0	6.0	1.3	1.0	0.049	55.5	7.00	0.220	53.6	1.000	3922	49.7	0.0	0.0
	3TR15	12.0	6.0	1.3	1.0	0.049	55.5	4.00	0.220	53.6	1.000	3922	61.7	0.0	0.0
	3TR30	12.0	6.0	1.3	1.0	0.049	55.5	2.00	0.220	53.6	1.000	3922	76.0	0.0	0.0
	4TR3	12.0	6.0	1.4	1.1	0.049	55.5	14.00	0.390	41.0	1.000	3922	35.0	0.0	0.0
	4TR7	12.0	6.0	1.4	1.1	0.049	55.5	7.00	0.390	41.0	1.000	3922	54.8	0.0	0.0
	4TR15	12.0	6.0	1.4	1.1	0.049	55.5	4.00	0.390	41.0	1.000	3922	74.0	0.0	0.0
	4TR30	12.0	6.0	1.4	1.1	0.049	55.5	2.00	0.390	41.0	1.000	3922	85.0	0.0	0.0
	5TR3	12.0	6.0	1.4	1.2	0.049	55.5	14.00	0.610	48.6	1.000	3922	43.0	0.0	0.0
	5TR7	12.0	6.0	1.4	1.2	0.049	55.5	7.00	0.610	48.6	1.000	3922	59.7	0.0	0.0
	5TR15	12.0	6.0	1.4	1.2	0.049	55.5	4.00	0.610	48.6	1.000	3922	76.5	0.0	0.0
	5TR30	12.0	6.0	1.4	1.2	0.049	55.5	2.00	0.610	48.6	1.000	3922	92.6	0.0	0.0
Cowan	R3	9.0	6.0	0.8	0.8	0.049	20.8	4.00	0.580	48.5	0.672	7699	71.8	0.0	0.0
	R5	9.0	6.0	0.8	0.8	0.049	20.8	4.00	0.580	48.5	0.672	8500	75.4	75.4	0.0
	R2	9.0	6.0	0.8	0.8	0.049	20.8	4.00	0.580	48.5	0.672	7299	79.0	158.0	0.0
	R1	9.0	6.0	0.8	0.8	0.049	20.8	4.00	0.580	48.5	0.672	7299	43.0	258.0	0.0
	S1	9.0	6.0	0.8	0.8	0.049	20.8	3.00	0.580	48.5	0.672	7199	82.6	206.5	0.0
	S4	9.0	6.0	0.8	0.8	0.049	20.8	3.00	0.580	48.5	0.672	6969	64.6	258.4	0.0
Gesund et al.	1	8.0	8.0	1.5	1.5	0.110	50.0	5.00	0.580	51.0	0.672	5029	79.0	79.0	0.0
	2	8.0	8.0	1.5	1.5	0.110	50.0	2.00	0.580	51.0	0.672	5299	102.0	102.0	0.0
	3	8.0	8.0	1.5	1.5	0.110	50.0	5.00	0.580	51.0	0.672	5309	61.0	122.0	0.0
	4	8.0	8.0	1.5	1.5	0.110	50.0	2.00	0.580	51.0	0.672	4679	67.0	134.0	0.0
	5	8.0	8.0	1.5	1.5	0.110	50.0	5.00	0.580	51.0	0.672	4239	49.0	147.0	0.0
	6	8.0	8.0	1.5	1.5	0.110	50.0	2.00	0.580	51.0	0.672	4059	56.0	168.0	0.0
	7	8.0	8.0	1.5	1.5	0.110	50.0	5.00	0.580	51.0	0.672	5279	43.0	173.0	0.0
	8	8.0	8.0	1.5	1.5	0.110	50.0	2.00	0.580	51.0	0.672	5739	44.0	176.0	0.0
	9	12.0	6.0	2.0	1.8	0.110	50.0	8.00	0.580	51.0	0.672	4859	60.0	120.0	0.0
	10	12.0	6.0	2.0	1.8	0.110	50.0	8.00	0.580	51.0	0.672	3899	44.0	176.0	0.0
	11	12.0	6.0	2.0	1.8	0.110	50.0	4.00	0.580	51.0	0.672	4859	68.0	138.0	0.0
	12	12.0	6.0	2.0	1.8	0.110	50.0	4.00	0.580	51.0	0.672	3899	53.0	213.0	0.0

APPENDIX B. EXPERIMENTAL DATA.

Investigator	Beam No.	Geometry (inches) h	b	a_1	a_2	Web Steel A_w sq. in.	f_w k.s.i.	s in.	Longitudinal Steel A_{L1} sq. in.	f_{L1} k.s.i.	R	f'_c p.s.i.	Failure Loads T kip. in.	M kip. in.	V kips
Lessig	BK1	11.8	7.9	1.0	1.2	0.122	49.1	4.90	0.560	34.5	1.000	2009	121.0	0.0	0.0
	BK1A	12.0	8.1	1.0	1.4	0.122	49.1	4.90	0.560	34.5	1.000	1569	104.0	0.0	0.0
	BK2	12.0	7.9	1.4	1.2	0.122	49.1	3.90	0.630	46.5	1.000	1709	147.8	0.0	0.0
	BK2A	11.8	7.9	1.2	1.2	0.122	49.1	3.90	0.630	46.5	1.000	1849	153.0	0.0	0.0
	BK3	15.8	6.7	1.0	1.4	0.122	49.1	4.90	0.630	46.5	1.000	1909	175.5	0.0	0.0
	BK3A	15.8	6.7	1.0	1.4	0.122	49.1	4.90	0.630	46.5	1.000	1569	149.3	0.0	0.0
	BU4	12.0	8.1	1.4	1.6	0.122	49.1	4.90	1.940	54.0	0.273	909	111.0	556.0	0.0
	BU4A	11.8	7.9	1.2	1.4	0.122	49.1	4.90	1.940	54.0	0.273	909	104.0	522.0	0.0
	BU5	11.8	7.9	1.2	1.4	0.122	49.1	4.90	0.990	49.7	1.000	679	69.5	86.8	0.0
	BU5A	11.8	7.9	1.0	1.4	0.122	49.1	4.90	0.990	49.7	1.000	714	67.6	174.0	0.0
	BU6	11.8	7.9	1.0	1.4	0.122	49.1	4.90	0.630	46.5	1.000	750	69.5	173.6	0.0
	BU6A	12.0	7.9	1.1	1.4	0.122	49.1	4.90	0.630	46.5	1.000	789	73.0	104.2	0.0
	BII7	12.0	6.1	1.4	1.4	0.122	40.0	3.90	1.900	55.5	0.163	1429	62.5	625.0	22.95
	BII7A	11.8	6.1	1.2	1.4	0.122	40.0	3.90	1.900	55.5	0.163	1659	62.5	625.0	22.95
	BII8	12.2	5.9	1.6	1.2	0.122	40.0	3.90	0.980	50.0	0.347	1509	74.0	382.0	14.50
	BII8A	11.8	5.9	1.2	1.3	0.122	40.0	3.90	0.980	50.0	0.347	1779	74.0	382.0	14.35
	BII9	11.8	6.1	1.2	1.3	0.122	40.0	3.90	0.980	50.0	0.347	1559	88.6	157.0	6.04
	BII9A	11.8	6.1	1.2	1.4	0.122	40.0	3.90	0.980	50.0	0.347	1729	94.0	156.0	6.07
	BII10	11.8	8.1	1.0	1.3	0.122	40.0	3.90	0.980	50.0	0.347	1639	137.0	348.0	12.99
	BII10A	11.8	7.9	1.2	1.2	0.122	40.0	3.90	0.980	50.0	0.347	1809	125.0	313.0	11.77
	BII11	12.0	7.9	1.2	1.1	0.122	40.0	3.90	0.980	50.0	0.347	1599	122.0	156.0	6.12
	BII11A	11.8	7.9	1.4	1.1	0.122	40.0	3.90	0.980	50.0	0.347	1659	115.0	156.0	6.14
	BII12	12.0	7.9	1.4	1.2	0.078	51.5	3.90	1.250	84.5	0.176	2779	146.0	243.0	9.24
	BII12A	12.0	7.9	1.2	1.2	0.078	51.5	3.90	1.250	84.5	0.176	2779	151.0	261.0	10.21
	BII13	11.8	6.1	1.2	1.4	0.121	45.5	3.90	1.900	55.5	0.189	2129	93.5	520.0	19.16
	BII13A	11.6	6.1	1.2	1.4	0.121	45.5	3.90	1.900	55.5	0.189	2579	125.0	625.0	23.16
	BII14	11.8	5.9	1.2	1.8	0.121	45.5	3.90	1.900	55.5	0.189	2379	83.2	416.0	15.51
	BII14A	12.0	5.9	1.2	2.0	0.121	45.5	3.90	1.900	55.5	0.189	2649	113.0	572.0	21.26
	BII15	12.2	7.9	1.2	1.6	0.121	45.5	3.90	0.980	49.5	0.418	2539	156.0	416.0	15.56
	BII15A	12.0	7.9	1.2	1.4	0.121	45.5	3.90	0.980	49.5	0.418	2759	151.0	416.0	15.52
	BII16	12.4	6.1	1.0	1.4	0.122	42.5	3.90	0.350	58.0	1.000	3259	92.0	156.0	4.07
	BII16A	12.0	6.1	1.0	1.4	0.122	42.5	3.90	0.350	58.0	1.000	3549	83.4	156.0	4.16

APPENDIX B. EXPERIMENTAL DATA

Investigator	Beam No.	Geometry (inches)				Web Steel			Longitudinal Steel				Failure Loads		
		h	b	a_1	a_2	A_w sq. in.	f_w k.s.i.	s in.	A_{L1} sq. in.	f_{L1} k.s.i.	R	f'_c p.s.i.	T kip. in.	M kip. in.	V kips
Lessig	BII17A	12.2	6.1	1.2	1.4	0.122	42.5	3.90	0.622	50.6	1.000	3969	90.4	313.0	8.06
	BII17	12.2	6.1	1.2	1.4	0.122	42.5	3.90	0.622	50.5	1.000	3799	83.4	278.0	7.16
	BII18	12.2	6.1	1.0	1.4	0.059	41.5	3.90	0.933	50.5	0.667	3509	114.5	191.0	4.99
	BII18A	12.2	6.1	1.0	1.2	0.059	41.5	3.90	0.933	50.5	0.667	3699	111.0	191.0	4.99
	BII19	12.0	6.1	0.8	1.4	0.059	41.5	3.90	0.622	50.5	1.000	3389	78.0	313.0	4.63
	BII19A	12.0	5.9	1.0	1.4	0.059	41.5	3.90	0.622	50.5	1.000	3589	79.0	313.0	4.71
	BII20	12.2	7.9	2.4	1.4	0.122	42.5	3.90	1.592	49.0	0.240	1489	125.0	313.0	11.77
	BII20A	12.0	7.9	2.0	1.4	0.122	42.5	3.90	1.592	49.0	0.240	1609	130.2	339.0	12.69
	BII21	12.0	7.9	2.0	1.9	0.122	42.5	3.90	1.592	49.0	0.240	1549	120.8	313.0	11.77
	BII21A	12.2	7.9	2.0	1.6	0.122	42.5	3.90	1.592	49.0	0.240	1629	99.0	313.0	11.78
	IIB	11.8	6.5	2.0	1.2	0.044	32.7	6.10	0.622	56.5	1.000	2409	41.7	206.5	10.20
	IIBA	11.8	7.1	2.3	1.2	0.044	32.7	6.10	0.622	56.5	1.000	2889	57.3	206.5	10.17
	WB	11.8	6.1	2.0	1.2	0.044	32.7	6.10	0.622	56.5	1.000	2409	53.0	132.0	6.64
	WBA	11.8	6.4	2.0	1.2	0.044	32.7	6.10	0.622	56.5	1.000	2889	57.3	143.0	7.16
Chinenkov	B28 0.1	11.8	7.9	1.4	1.4	0.060	41.0	3.20	0.995	54.0	1.000	1223	48.6	486.0	0.0
	B28 0.1A	12.0	7.9	1.4	1.4	0.060	41.0	3.20	0.995	52.5	1.000	1223	46.9	469.0	0.0
	B28 0.2	12.2	7.9	1.4	1.4	0.060	41.0	3.20	1.030	52.3	1.000	1023	83.4	417.0	0.0
	B28 0.2A	12.0	7.9	1.6	1.4	0.060	41.0	3.20	0.995	53.3	1.000	1223	83.4	417.0	0.0
	B28 0.4	12.0	7.9	1.6	1.4	0.060	41.0	3.20	1.050	52.3	1.000	2783	146.0	365.0	0.0
	B28 0.4A	12.0	7.9	1.6	1.4	0.060	41.0	3.20	1.050	52.3	1.000	2783	139.0	347.0	0.0
	B28 0.4B	12.0	7.9	1.6	1.4	0.060	41.0	3.20	1.040	52.3	1.000	4319	146.0	365.0	0.0
	B28 0.4C	11.8	7.9	1.3	1.4	0.060	41.0	3.20	1.080	55.0	1.000	4319	153.0	382.0	0.0
	B28 0.4D	11.8	7.9	1.4	1.4	0.060	41.0	3.20	1.010	52.5	1.000	2271	125.0	313.0	0.0
	B28 0.4E	12.0	7.9	1.4	1.4	0.060	40.0	3.20	1.030	52.5	1.000	2271	132.0	330.0	0.0
	B28 0.4F	11.8	7.9	1.4	1.4	0.121	40.0	3.20	1.050	53.0	1.000	2271	139.0	347.0	0.0

Invest-igator	Beam No.	Geometry (inches) h	b	a_1	a_2	Web Steel A_w sq. in.	f_w k.s.i.	s in.	Longitudinal Steel A_{L1} sq. in.	f_{L1} k.s.i.	R	f'_c p.s.i.	Failure Loads T kip. in.	M kip. in.	V kips
Lyalin	B8 K	12.2	7.9	1.4	1.4	0.079	44.5	3.20	1.010	54.5	1.000	1359	125.0	0.0	0.0
	B8 KA	12.2	7.9	1.4	1.4	0.079	44.5	3.20	1.010	54.5	1.000	2399	153.0	0.0	0.0
	B8 0.1	12.2	7.9	1.4	1.4	0.079	44.5	3.20	0.980	54.5	1.000	1759	52.0	520.0	12.52
	B8 0.1A	12.2	7.9	1.4	1.4	0.079	44.5	3.20	1.030	55.8	1.000	1759	55.5	555.0	13.36
	B8 0.2	11.8	7.9	1.4	1.4	0.079	44.5	3.20	1.000	56.2	1.000	1535	90.0	451.0	10.85
	B8 0.2A	12.2	7.9	1.4	1.4	0.079	44.5	3.20	1.040	52.0	1.000	1759	97.0	486.0	11.69
	B8 0.4	12.2	7.9	1.4	1.4	0.079	44.5	3.20	1.040	56.4	1.000	1727	132.0	347.0	8.42
	B8 0.4A	12.0	7.9	1.4	1.4	0.079	44.5	3.20	1.030	57.6	1.000	2223	139.0	347.0	8.38
	B7 0.2	12.0	7.9	1.4	1.4	0.059	41.6	3.20	1.040	55.7	1.000	1535	93.8	468.0	11.30
	B7 0.2A	12.0	7.9	1.4	1.4	0.059	41.6	3.20	1.010	55.5	1.000	1927	90.2	451.0	10.87
	B10 0.2	12.0	7.9	1.4	1.4	0.117	40.8	3.20	1.020	52.5	1.000	2215	104.0	521.0	12.53
	B10 0.2A	12.2	7.9	1.4	1.4	0.117	40.8	3.20	1.020	53.0	1.000	2399	104.0	521.0	12.53
	B1	7.9	8.7	1.4	1.4	0.074	63.5	3.20	1.440	53.3	1.000	4087	90.3	452.0	10.82
	B1A	7.9	8.7	1.4	1.4	0.074	63.5	3.20	1.490	55.0	1.000	3935	90.3	452.0	12.75
	B2	11.8	8.7	1.4	1.4	0.074	63.5	3.20	1.490	47.8	1.000	4087	139.0	694.0	16.65
	B2A	11.8	8.7	1.4	1.4	0.074	63.5	3.20	1.460	53.0	1.000	4095	139.0	694.0	16.65
	B3	11.8	8.7	1.4	1.4	0.074	63.5	3.20	1.440	53.0	1.000	3927	194.0	486.0	17.48
	B3A	11.8	8.7	1.4	1.4	0.074	63.5	3.20	1.490	51.0	1.000	3927	194.0	486.0	17.48
	B5	15.7	8.7	1.4	1.4	0.074	63.5	3.20	1.460	54.6	1.000	3703	194.0	972.0	23.24
	B5A	15.7	8.7	1.4	1.4	0.074	63.5	3.20	1.480	48.8	1.000	4135	194.0	972.0	23.24
	B6	15.7	6.7	1.4	1.4	0.074	63.5	3.20	1.460	47.4	1.000	4215	167.0	833.0	20.19
	B6A	15.7	6.7	1.4	1.4	0.074	63.5	3.20	1.460	47.0	1.000	4135	181.0	903.0	21.88
	B7	11.8	7.9	1.4	1.4	0.270	37.0	4.10	1.060	47.0	1.000	2655	221.0	0.0	0.0
	B7A	11.8	7.5	1.2	1.2	0.270	37.0	4.10	1.060	47.0	1.000	2655	208.0	0.0	0.0
	B8	11.8	7.5	1.2	1.2	0.070	68.7	4.10	1.070	49.2	1.000	2591	156.0	156.0	3.83
	B8A	11.8	7.9	1.4	1.4	0.070	68.7	4.10	1.060	47.0	1.000	2615	156.0	156.0	3.63
	B9	9.1	5.9	1.4	1.4	0.075	50.7	4.10	1.000	50.8	1.000	1023	39.0	208.0	10.05
	B9A	9.1	5.9	1.4	1.4	0.075	50.7	4.10	1.070	52.6	1.000	1087	47.0	235.0	11.39
	B10	9.1	7.1	1.4	1.4	0.075	50.7	4.10	1.060	47.0	1.000	967	36.4	182.0	8.82
	B10A	9.1	5.9	1.4	1.4	0.075	50.7	4.10	1.060	47.0	1.000	1031	41.8	209.0	10.13
	B11	9.1	5.9	1.4	1.4	0.075	50.7	4.10	0.707	57.3	1.000	1055	36.6	209.0	10.10
	B11A	9.1	5.9	1.4	1.4	0.075	50.7	4.10	0.670	54.7	1.000	1119	36.6	209.0	10.10
	B12	9.1	5.9	1.4	1.4	0.075	50.7	4.10	0.682	56.0	1.000	887	24.2	78.0	3.78
	B12A	9.1	5.9	1.4	1.4	0.075	50.7	4.10	0.682	56.0	1.000	927	31.2	104.0	5.07

APPENDIX B. EXPERIMENTAL DATA

Investigator	Beam No.	Geometry (inches) h	b	a_1	a_2	Web Steel A_w sq. in.	f_w k.s.i.	s in.	Longitudinal Steel A_{L1} sq. in.	f_{L1} k.s.i.	R	f'_c p.s.i.	Failure Loads T kip. in.	M kip. in.	V kips
Yudin	1	6.3	3.5	1.0	1.0	0.031	64.0	5.90	0.244	49.8	0.422	1351	7.2	36.2	1.84
	2	6.3	3.5	1.0	1.0	0.031	64.0	5.90	0.244	49.8	0.422	1351	7.2	36.2	1.84
	6	6.3	3.5	1.0	1.0	0.031	64.0	5.90	0.244	49.8	0.422	1351	4.6	45.7	2.32
	7	6.3	3.5	1.0	1.0	0.031	64.0	5.90	0.244	49.8	0.422	1351	9.9	49.3	2.51
	10	6.3	3.5	1.0	1.0	0.031	64.0	5.90	0.244	49.8	0.422	1351	5.9	59.0	3.00
	3	6.3	3.5	1.0	1.0	0.031	64.0	2.95	0.244	49.8	0.422	1351	11.9	59.4	3.02
	4	6.3	3.5	1.0	1.0	0.031	64.0	2.95	0.244	49.8	0.422	1351	11.9	59.4	3.02
	8	6.3	3.5	1.0	1.0	0.031	64.0	2.95	0.244	49.8	0.422	1351	13.1	65.3	3.32
	9	6.3	3.5	1.0	1.0	0.031	64.0	2.95	0.244	49.8	0.422	1351	13.1	65.3	3.32
	11	6.3	3.5	1.2	1.0	0.031	64.0	3.94	0.312	49.8	1.000	2319	14.3	71.7	3.65
	12	6.3	3.5	1.2	1.0	0.031	64.0	3.94	0.312	49.8	1.000	2319	11.1	55.7	2.83
	13	6.3	3.5	1.2	1.0	0.031	64.0	3.94	0.312	49.8	1.000	2319	11.1	55.7	2.83
	17	6.3	3.5	1.2	1.0	0.031	64.0	7.87	0.312	49.8	0.500	2175	7.2	36.0	1.83
	18	6.3	3.5	1.2	1.0	0.031	64.0	7.87	0.312	49.8	0.500	2175	7.9	39.4	2.00
	19	6.3	3.5	1.2	1.0	0.031	64.0	7.87	0.312	49.8	0.500	2175	7.9	39.4	2.00
	20	6.3	3.5	1.2	1.0	0.031	64.0	7.87	0.312	49.8	0.500	2175	9.1	45.7	2.32
	21	6.3	3.5	1.2	1.0	0.031	64.0	7.87	0.312	49.8	0.500	2175	7.9	39.4	2.00
	22	6.3	3.5	1.0	1.0	0.031	64.0	7.87	0.156	49.8	2.000	2175	6.5	32.6	1.66
Evans and Sarkar	HB1	7.6	6.1	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	6919	44.1	0.0	0.0
	HB2	7.6	6.1	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	6463	33.9	66.8	0.0
	HB3	7.5	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	6359	20.4	75.3	0.0
	HB4	7.7	6.1	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	7000	15.7	81.6	0.0
	HB5	7.5	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	6063	13.2	81.5	0.0
	HB7	9.0	6.0	0.8	0.8	0.049	40.8	4.00	0.220	54.5	0.895	5119	36.1	0.0	0.0
	HB8	9.0	6.0	0.8	0.8	0.049	40.8	4.00	0.220	54.5	0.895	5099	21.4	79.6	0.0
	HB9	9.0	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	4031	18.3	85.1	0.0
	HB10	9.1	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	6439	17.3	91.3	0.0
	HB11	9.0	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	5199	14.1	94.0	0.0
	HB13	12.0	6.0	0.8	0.8	0.049	40.8	4.00	0.220	54.5	0.895	5159	51.3	0.0	0.0
	HB14	12.1	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	5199	41.7	82.1	0.0
	HB15	12.0	6.0	0.8	0.8	0.049	40.8	4.00	0.220	54.5	0.895	5159	29.9	111.0	0.0
	HB16	12.1	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	4031	23.5	129.0	0.0
	HB17	12.0	6.0	0.8	0.8	0.049	46.0	4.00	0.220	59.0	0.759	6439	19.4	137.0	0.0

APPENDIX C

Optimum Value of r

The total volume of reinforcement per unit length of beam is given by

$$W = (A_{L1} + RA_{L1}) + \frac{A_w}{s} \, 2\left[(h-2a_3) + 2(b-2a_4)\right] \quad \ldots(C1)$$
$$= C_1 A_{L1}(1 + kr)$$

where

$$k = \frac{f_{L1}}{0.8f_w} \; \frac{2}{(1+R)b} \left[(h-2a_3) + (b-2a_4)\right] \quad \ldots(C2)$$

and

$$C_1 = 1 + R$$

For a given size of beam both C_1 and k are constant if the ratio of top to bottom steel remains constant. The area of steel A_{L1} to prevent the most common type of failure, mode 1, is, from equation (5·7),

$$A_{L1} = T_1/(h-a_1-x_1) \frac{2r}{1+2\alpha} \sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{1+2\alpha}{r}} - \frac{1}{\psi}$$

When this is substituted for A_{L1}, equation (C1) becomes

$$W = C_2 \frac{1 + kr}{r\left[\sqrt{\left(\frac{1}{\psi}\right)^2 + \frac{1+2\alpha}{r}} - \frac{1}{\psi}\right]} \quad \ldots(C3)$$

This function is a minimum when

$$r = \cfrac{1}{k + \cfrac{2\sqrt{k}}{\psi\sqrt{1+2\alpha}}} \qquad \ldots C4)$$

The value of k (see equation C2) is dependent upon α, R and the ratio of cover on the steel to the width of the section. Hence, the influence of these variables would need to be considered in any attempt to find a minimum weight solution. An examination of these variables for practical situations indicates that k might vary between 2 and 7. Test, reference (2.18), (2.14), show, however, that reinforced beams behave in a relatively ductile manner when subjected to torsion and bending if r is not unduly small. It is therefore advantageous to adopt a k value somewhat less than the upper limit.

The design process is, of course, considerably simplified if a constant value is adopted for k. In view of the above remarks the value k = 4 has been adopted so that

$$r_o = \cfrac{1}{4 + \cfrac{4}{\psi\sqrt{1 + 2\alpha}}}$$

APPENDIX D

The experimental data listed in Appendix B has been analysed, and the results of the analysis are presented in this Appendix. The failure loads of each beam have been expressed in terms of the three ratios T/T_o, M/M_u and V/V_o. T_o is the pure torsional strength of the beam as calculated from the theory set out in Chapter 5 (see Equation 5.11). M_u is the calculated flexural strength and V_o is the shear capacity of the beam as given by the A.C.I. code.

The parameters r/r_o, pf_y/f'_c and $V_{eff}/bd\sqrt{f'_c}$, which are listed in the table, are related to the restrictions on the theory discussed in Chapter 6.

i.e. $r/r_o \quad 0.9$ ... (D.1)

$pf_y/f'_c \quad 0.40$ ... (D.2)

$V_{eff}/bd\sqrt{f'_c} \quad 8$ ... (D.3)

Beams which do not satisfy the above requirements have been included in the table, but they have been marked with the letters R, P and V if they violate the limits set out in equations (D.1), (D.2) and (D.3) respectively.

For each beam the theoretical failure torque has been calculated in four ways, from the Modes 1, 2 and 3 equations and from the effective shear formula. These values have been expressed in the table as the ratios $T_{exp}/T_{theor.}$ Also listed in the table is the critical value, as $T_{exp}/T_{theor.}$ and its associated mode.

APPENDIX D. ANALYSIS OF TEST DATA

Investigator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd\sqrt{f'_c}}$	$\frac{T_{exp}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Critical Mode	Restrictions
This Investigation	RE1	0.83	0.05	0.01	2.12	0.00	6.2	0.85	0.88	0.82	0.68	0.88	2	
	RE2	0.89	0.23	0.03	2.23	0.00	5.8	1.01	0.90	0.85	0.72	1.01	1	
	RE3	0.83	0.32	0.04	2.60	0.00	5.7	1.01	0.85	0.79	0.72	1.01	1	
	RE4	0.76	0.61	0.11	3.19	0.00	5.2	1.12	0.85	0.65	0.73	1.12	1	
	RE5	0.67	0.78	0.10	3.70	0.00	4.5	1.16	0.76	0.57	0.65	1.16	1	
	RE4*	0.39	0.96	0.13	5.61	0.00	3.2	1.10	0.50	0.26	0.45	1.10	1	
	RU1	0.60	0.02	0.01	0.72	0.16	4.6	0.61	0.75	1.22	0.70	1.22	3V	R
	RU3A*	0.60	0.02	0.01	0.72	0.13	4.2	0.62	0.75	1.23	0.65	1.23	3V	R
	RU2	0.70	0.17	0.0	0.99	0.16	5.8	0.79	0.86	1.11	0.81	1.11	3	
	RU3	0.86	0.28	0.0	0.97	0.16	6.3	1.01	1.06	1.26	1.00	1.26	3	
	RU3A	0.71	0.49	0.0	1.27	0.13	6.1	1.00	0.88	0.76	0.76	1.00	1	
	RU4	0.70	0.49	0.16	1.28	0.16	6.4	0.99	1.00	1.21	0.97	1.21	3V	
	RU5	0.62	0.84	0.28	1.83	0.16	6.8	1.17	1.00	0.87	0.99	1.17	1	
	RU5A	0.55	0.88	0.28	2.04	0.13	6.1	1.14	0.93	0.70	0.87	1.14	1	
	RU6	0.48	0.95	0.31	2.33	0.16	6.0	1.15	0.86	0.56	0.87	1.15	1	
	36T4	0.57	0.94	0.28	2.19	0.10	5.3	1.21	0.96	0.64	0.84	1.21	1	
	36T4C	0.87	0.24	0.0	0.96	0.10	5.2	1.00	1.07	1.23	0.87	1.23	3	
	36T5.5	0.93	0.67	0.0	1.06	0.09	4.3	1.32	1.15	0.91	0.98	1.32	1	
	77T5	0.83	0.78	0.0	1.07	0.00	4.7	1.31	0.82	0.52	0.99	1.31	1	
	77T4	0.87	0.67	0.0	1.12	0.00	5.7	1.26	0.86	0.60	1.00	1.26	1	
	24T3	0.73	0.30	0.0	2.13	0.05	5.1	0.90	0.92	0.86	0.58	0.92	2	
	38T5	0.70	0.52	0.29	0.71	0.25	6.1	1.01	1.21	1.29	1.15	1.29	3V	R

Investigator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd\sqrt{f'_c}}$	$\frac{T_{exp}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Mode	Restrictions
This Investigation	R4.20	0.61	0.82	0.0	1.01	0.32	6.7	1.15	0.79	0.42	0.96	1.15	1	
	R4.24	0.54	0.65	0.0	1.04	0.37	8.1	0.96	0.69	0.40	0.97	0.97	VEF V	
	R3.20	0.62	0.73	0.0	0.91	0.27	6.0	1.08	0.78	0.46	0.81	1.08	1	
	R3.24	0.61	0.66	0.0	0.94	0.29	7.0	1.02	0.77	0.48	0.90	1.02	1	
	R3.30	0.65	0.60	0.0	0.93	0.27	7.8	1.02	0.82	0.59	0.99	1.02	1	
	R2.24	0.71	0.84	0.0	1.00	0.18	4.9	1.24	0.84	0.50	0.84	1.24	1	
	R2.30	0.74	0.71	0.0	0.97	0.18	5.7	1.18	0.88	0.61	0.85	1.18	1	
	R2.38	0.73	0.56	0.0	0.94	0.17	6.7	1.06	0.87	0.69	0.88	1.06	1	
	R1.30	0.67	0.65	0.0	0.97	0.13	5.0	1.07	0.83	0.54	0.76	1.07	1	
	R4A	0.58	0.23	0.21	0.43	0.38	8.0	0.70	0.89	1.09	1.09	1.09	3 V	R
	R4B	0.59	0.41	0.37	0.57	0.38	9.0	0.83	1.01	0.92	1.27	1.27	VEF V	R
	R4.20A	0.58	0.55	0.45	1.13	0.32	13.1	0.91	0.98	0.93	1.60	1.60	VEF V	
	R4.20B	0.57	0.37	0.31	0.93	0.32	11.2	0.79	0.90	1.04	1.44	1.44	VEF V	
	R4.24A	0.56	0.63	0.44	1.07	0.37	12.5	0.96	0.96	0.88	1.50	1.50	VEF V	
	R3A	0.61	0.27	0.21	0.63	0.31	8.9	0.76	0.90	1.10	1.22	1.22	VEF V	R
	R3B	0.55	0.48	0.38	0.90	0.31	11.1	0.84	0.92	0.79	1.27	1.27	VEF V	
	R3.20A	0.67	0.53	0.38	0.82	0.27	9.9	0.98	1.11	1.01	1.37	1.37	VEF V	R
	R3.20B	0.64	0.23	0.17	0.56	0.27	7.5	0.76	0.93	1.19	1.12	1.19	3 V	R
	R3.24A	0.55	0.27	0.21	0.77	0.29	9.4	0.70	0.81	0.98	1.15	1.15	VEF V	R
	R3.24B	0.53	0.46	0.34	1.01	0.29	11.6	0.81	0.87	0.81	1.26	1.26	VEF V	
	R3.30A	0.54	0.59	0.42	1.36	0.27	12.5	0.91	0.91	0.78	1.42	1.42	VEF V	
	R3.30B	0.55	0.25	0.19	0.89	0.27	9.5	0.69	0.80	1.02	1.21	1.21	VEF V	R
	R2A	0.79	0.45	0.25	0.78	0.19	8.4	1.05	1.14	1.14	1.21	1.21	VEF V	R
	R2B	0.72	0.32	0.18	0.70	0.19	7.7	0.90	1.01	1.10	1.06	1.10	3 V	R
	R2.24A	0.93	0.57	0.31	0.75	0.18	8.1	1.25	1.36	1.30	1.37	1.37	VEF V	R
	R2.30A	0.63	0.42	0.23	1.00	0.18	9.4	0.88	0.92	0.90	1.08	1.08	VEF V	
	R2.30B	0.64	0.29	0.16	0.85	0.18	8.3	0.80	0.88	0.99	1.02	1.02	VEF V	R
	R2.38A	0.62	0.48	0.26	1.24	0.17	10.1	0.91	0.92	0.87	1.16	1.16	VEF V	
	R2.38B	0.56	0.28	0.16	1.03	0.17	9.0	0.72	0.77	0.86	0.96	0.96	VEF V	
	R1.30A	0.61	0.43	0.21	0.96	0.13	7.5	0.86	0.94	0.77	0.89	0.94	2	
	R1.30B	0.59	0.27	0.14	0.79	0.13	7.4	0.74	0.85	0.82	0.79	0.85	2	R

APPENDIX D. ANALYSIS OF TEST DATA

Investigator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd/f'_c}$	$\frac{T_{exp.}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Critical Mode	Restrictions
Collins	V1	0.30	0.65	1.15	0.77	0.29	4.9	0.76	1.04	0.13	1.64	1.64	VEF	R
	V1*	0.47	0.50	0.89	0.43	0.29	4.9	0.78	1.10	0.36	1.67	1.67	VEF	R
	V2	0.41	0.49	0.88	0.47	0.29	4.9	0.72	1.01	0.28	1.54	1.54	VEF	R
	V2*	0.67	0.20	0.37	0.19	0.29	5.0	0.77	1.03	0.90	1.46	1.46	VEF	R
	V3	0.13	0.82	1.44	2.02	0.29	4.9	0.84	1.01	0.02	1.65	1.65	VEF	
	V3*	0.61	0.47	0.84	0.33	0.29	4.9	0.89	1.24	0.57	1.85	1.85	VEF	R
	V4	0.71	0.29	0.53	0.22	0.29	4.9	0.87	1.17	0.87	1.69	1.69	VEF	R
	V4*	0.31	0.79	1.40	0.88	0.29	4.9	0.90	1.21	0.12	1.91	1.91	VEF	R
	V5*	0.70	0.0	0.0	0.10	0.29	4.5	0.70	0.84	1.21	1.24	1.24	VEF	R
	V6	0.19	0.80	1.41	1.38	0.29	4.9	0.84	1.07	0.05	1.72	1.72	VEF	
	V6*	0.64	0.41	0.74	0.30	0.29	4.9	0.88	1.22	0.65	1.80	1.80	VEF	R
	V7	0.65	0.35	0.49	0.26	0.29	4.8	0.85	1.07	0.72	1.58	1.58	VEF	R
	V7*	0.80	0.25	0.61	0.19	0.29	5.1	0.94	1.34	1.07	1.88	1.88	VEF	R
	U1	0.66	0.0	0.0	0.15	0.29	5.9	0.66	0.80	1.14	1.11	1.14	3	R
	U1*	0.61	0.24	0.33	0.30	0.29	6.5	0.74	0.95	0.77	1.30	1.30	VEF	R
	U2	0.27	0.82	1.11	1.33	0.29	6.5	0.90	1.01	0.09	1.53	1.53	VEF	
	U2*	0.53	0.38	0.52	0.43	0.29	6.5	0.75	0.96	0.51	1.34	1.34	VEF	R
	U3	0.51	0.63	0.86	0.64	0.29	6.5	0.92	1.15	0.35	1.66	1.66	VEF	R
	U3*	0.41	0.86	1.16	0.97	0.29	6.5	1.02	1.21	0.18	1.80	1.80	VEF	
	T1	0.65	0.69	0.59	0.77	0.18	5.3	1.08	1.50	0.64	1.39	1.50	2	R
	T2	0.80	0.22	0.18	0.38	0.18	4.1	0.92	1.34	1.63	1.18	1.63	3V	R
	T4	0.46	1.11	0.93	1.44	0.18	5.5	1.27	1.54	0.18	1.50	1.54	2	
	T4*	0.70	0.78	0.66	0.79	0.18	5.3	1.19	1.64	0.66	1.53	1.64	2	R
	T5	0.55	0.92	0.77	1.07	0.18	5.4	1.17	1.52	0.31	1.45	1.52	2	
	T5*	0.65	0.0	0.0	0.24	0.18	3.0	0.65	0.95	1.55	0.86	1.55	3	R
	T6	0.72	0.42	0.36	0.53	0.18	5.1	0.96	1.38	1.15	1.25	1.38	2	R
	T6*	0.25	1.24	1.04	2.65	0.18	5.7	1.29	1.35	0.05	1.36	1.36	VEF	

APPENDIX D. ANALYSIS OF TEST DATA

Investigator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd\sqrt{f'_c}}$	$\frac{T_{exp.}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Critical Mode	Restrictions
Ernst	3TR3	1.16	0.0	0.0	0.29	0.00	2.0	1.16	1.11	1.16	0.77	1.16	1	R
	3TR7	1.19	0.0	0.0	0.58	0.00	2.8	1.19	1.14	1.19	0.83	1.19	1	R
	3TR15	1.12	0.0	0.0	1.01	0.00	3.7	1.12	1.07	1.12	0.74	1.12	1	
	3TR30	0.97	0.0	0.0	2.03	0.00	5.2	0.97	0.93	0.97	0.63	0.97	1	
	4TR3	1.04	0.0	0.0	0.21	0.00	2.3	1.04	1.00	1.04	0.80	1.04	1	
	4TR7	1.15	0.0	0.0	0.43	0.00	3.2	1.15	1.11	1.15	0.92	1.15	1	R
	4TR15	1.17	0.0	0.0	0.75	0.00	4.2	1.17	1.13	1.17	0.90	1.17	1	R
	4TR30	0.95	0.0	0.0	1.50	0.00	6.0	0.95	0.92	0.95	0.71	0.95	1	
	5TR3	0.97	0.0	0.0	0.12	0.00	2.9	0.97	0.96	0.97	0.98	0.98	VEF	R
	5TR7	0.95	0.0	0.0	0.23	0.00	4.0	0.95	0.94	0.95	1.01	1.01	VEF	R
	5TR15	0.92	0.0	0.0	0.40	0.00	5.5	0.92	0.91	0.92	0.93	0.93	VEF	R
	5TR30	0.79	0.0	0.0	0.81	0.00	7.9	0.79	0.78	0.79	0.77	0.79	1	R
Cowan	R3	1.50	0.0	0.0	0.19	0.02	2.4	1.50	1.61	1.83	1.54	1.83	3	R
	R5	1.57	0.34	0.0	0.28	0.02	2.5	1.75	1.69	1.68	1.57	1.75	1	R
	R2	1.65	0.72	0.0	0.38	0.03	2.4	2.05	1.78	1.55	1.73	2.05	1	R
	R1	0.90	1.17	0.0	0.75	0.03	1.6	1.66	0.97	0.53	0.94	1.66	1	R
	S1	1.50	0.94	0.0	0.57	0.03	2.6	2.04	1.61	1.26	1.63	2.04	1	R
	S4	1.17	1.18	0.0	0.75	0.03	2.2	1.90	1.26	0.80	1.29	1.90	1	R
Gesund et al.	1	0.82	0.44	0.0	1.36	0.04	4.0	1.07	0.90	0.73	0.74	1.07	1	
	2	0.67	0.57	0.0	3.40	0.04	5.3	1.01	0.73	0.50	0.67	1.01	1	
	3	0.63	0.68	0.0	1.86	0.04	3.1	1.06	0.69	0.42	0.57	1.06	1	
	4	0.44	0.75	0.0	4.65	0.04	3.9	0.96	0.48	0.22	0.47	0.96	1	
	5	0.52	0.83	0.0	2.36	0.04	2.7	1.08	0.56	0.26	0.47	1.08	1	
	6	0.37	0.95	0.0	5.89	0.05	3.1	1.08	0.41	0.13	0.42	1.08	1	
	7	0.45	0.96	0.0	2.87	0.04	2.0	1.14	0.49	0.18	0.40	1.14	1	
	8	0.29	0.97	0.0	7.14	0.03	2.1	1.05	0.31	0.08	0.28	1.05	1	
	9	0.86	0.43	0.0	0.60	0.03	3.5	1.10	1.00	0.77	0.74	1.10	1	R
	10	0.64	0.64	0.0	0.88	0.04	3.0	1.04	0.74	0.44	0.56	1.04	1	R
	11	0.69	0.50	0.0	1.21	0.03	4.4	0.98	0.80	0.55	0.54	0.98	1	
	12	0.54	0.78	0.0	1.77	0.04	3.6	1.06	0.63	0.30	0.47	1.06	1	

APPENDIX D. ANALYSIS OF TEST DATA

Investigator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd\sqrt{f'_c}}$	$\frac{T_{exp.}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Critical Mode	Restrictions
Lessig	BK1	0.97	0.0	0.0	1.65	0.00	6.6	0.97	0.98	0.97	0.80	0.98	2	
	BK1A	0.84	0.0	0.0	1.59	0.00	6.7	0.84	0.87	0.84	0.73	0.87	2	
	BK2	0.93	0.0	0.0	1.36	0.00	9.3	0.93	0.91	0.93	1.08	1.08	VEF V	
	BK2A	0.95	0.0	0.0	1.36	0.00	9.1	0.95	0.94	0.95	1.08	1.08	VEF V	
	BK3	1.15	0.0	0.0	0.82	0.00	8.2	1.15	1.19	1.15	1.21	1.21	VEF V	R
	BK3A	0.98	0.0	0.0	0.82	0.00	9.0	0.98	1.01	0.98	1.13	1.13	VEF V	R
	BU4	0.76	1.03	0.0	1.03	0.97	5.9	1.43	0.98	0.49	1.06	1.43	1	P
	BU4A	0.70	0.97	0.0	1.07	1.00	6.2	1.34	0.90	0.45	1.04	1.34	1	P
	BU5	0.38	0.19	0.0	1.02	0.00	13.3	0.48	0.39	0.30	0.81	0.81	VEF V	
	BU5A	0.35	0.36	0.0	1.43	0.00	10.4	0.58	0.37	0.22	0.75	0.75	VEF V	
	BU6	0.48	0.60	0.0	2.29	0.00	6.9	0.87	0.50	0.26	0.75	0.87	1	
	BU6A	0.51	0.36	0.0	1.74	0.00	8.5	0.72	0.52	0.36	0.76	0.76	VEF V	
	BII7	0.61	1.29	1.17	1.23	0.95	9.3	1.53	1.73	0.28	2.01	2.01	VEF V	P
	BII7A	0.54	1.16	1.09	1.23	0.82	9.5	1.37	1.57	0.24	1.87	1.87	VEF V	P
	BII8	0.55	0.87	0.75	1.57	0.34	12.6	1.13	1.02	0.39	1.78	1.78	VEF V	
	BII8A	0.53	0.83	0.68	1.52	0.29	12.0	1.09	1.05	0.38	1.63	1.63	VEF V	
	BII9	0.61	0.34	0.30	0.87	0.32	12.4	0.81	0.93	0.82	1.43	1.43	VEF V	R
	BII9A	0.66	0.34	0.28	0.81	0.29	11.3	0.85	1.02	0.91	1.43	1.43	VEF V	R
	BII10	0.68	0.72	0.46	1.56	0.22	9.6	1.13	1.18	0.71	1.41	1.41	VEF V	
	BII10A	0.66	0.68	0.41	1.54	0.21	9.6	1.08	1.09	0.70	1.30	1.30	VEF V	
	BII11	0.62	0.33	0.22	1.15	0.23	10.1	0.81	0.88	0.85	1.13	1.13	VEF V	
	BII11A	0.62	0.35	0.23	1.18	0.23	10.0	0.82	0.88	0.83	1.10	1.10	VEF V	
	BII12	0.60	0.25	0.29	0.46	0.37	7.1	0.74	0.95	1.07	1.23	1.23	VEF	R
	BII12A	0.59	0.26	0.32	0.47	0.37	7.1	0.74	0.97	1.04	1.27	1.27	VEF	R
	BII13	0.62	0.77	0.80	0.85	0.62	10.6	1.12	1.38	0.54	1.83	1.83	VEF V	R P
	BII13A	0.74	0.83	0.90	0.79	0.52	11.0	1.27	1.58	0.74	2.17	2.17	VEF V	R P
	BII14	0.60	0.58	0.64	0.61	0.57	8.9	0.96	1.40	0.62	1.56	1.56	VEF V	R P
	BII14A	0.78	0.72	0.81	0.54	0.51	8.3	1.22	1.91	0.80	1.98	1.98	VEF V	R P
	BII15	0.80	0.88	0.44	1.59	0.13	7.7	1.35	1.40	0.75	1.35	1.40	2	
	BII15A	0.76	0.89	0.43	1.74	0.12	7.7	1.33	1.30	0.70	1.29	1.33	1	
	BII16	0.89	0.71	0.13	1.98	0.00	5.4	1.31	1.08	0.77	0.89	1.31	1	
	BII16A	0.82	0.74	0.13	2.08	0.00	5.1	1.27	1.01	0.70	0.81	1.27	1	

APPENDIX D. ANALYSIS OF TEST DATA

Invest-igator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd/\overline{f'_c}}$	$\frac{T_{exp.}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Critical Mode	Restrict-ions
Lessig	BII17A	0.74	0.97	0.24	1.85	0.00	5.5	1.37	0.98	0.54	0.94	1.37	1	
	BII17	0.68	0.87	0.22	1.81	0.00	5.7	1.24	0.90	0.50	0.88	1.24	1	
	BII18	1.11	0.40	0.22	0.40	0.07	5.5	1.33	1.47	1.18	1.56	1.56	VEF	R
	BII18A	1.03	0.40	0.22	0.44	0.06	5.5	1.25	1.31	1.09	1.51	1.51	VEF	R
	BII19	0.90	0.96	0.21	0.96	0.00	4.2	1.50	1.16	0.74	1.15	1.50	1	
	BII19A	0.98	0.98	0.22	0.90	0.00	4.2	1.59	1.23	0.80	1.22	1.59	1	R
	BII20	0.75	0.61	0.49	0.96	0.51	10.5	1.11	1.19	0.96	1.55	1.55	VEF V	P
	BII20A	0.70	0.59	0.50	0.99	0.47	10.6	1.05	1.16	0.87	1.54	1.54	VEF V	P
	BII21	0.72	0.56	0.47	0.83	0.48	8.9	1.06	1.30	0.91	1.46	1.46	VEF V	R P
	BII21A	0.49	0.49	0.45	1.03	0.45	10.7	0.80	0.90	0.53	1.22	1.22	VEF V	P
	IIB	0.81	0.69	0.90	0.45	0.00	3.6	1.23	1.35	0.54	1.81	1.81	VEF	R
	IIBA	1.02	0.69	0.84	0.42	0.00	3.3	1.42	1.48	0.73	1.91	1.91	VEF	R
	WB	1.12	0.45	0.60	0.27	0.00	3.7	1.36	1.43	0.91	1.87	1.87	VEF	R
	WBA	1.11	0.47	0.61	0.29	0.00	3.5	1.37	1.44	0.90	1.82	1.82	VEF	R
Chinenkov	B28 0.1	0.36	1.01	0.0	1.82	0.00	3.0	1.12	0.37	0.12	0.46	1.12	1	
	B28 0.1A	0.35	0.98	0.0	1.86	0.00	3.0	1.09	0.35	0.11	0.43	1.09	1	
	B28 0.2	0.61	0.82	0.0	1.05	0.00	5.4	1.14	0.61	0.32	0.77	1.14	1	
	B28 0.2A	0.65	0.89	0.0	1.07	0.00	4.8	1.23	0.64	0.34	0.79	1.23	1	
	B28 0.4	1.10	0.74	0.0	0.67	0.00	4.5	1.53	1.08	0.79	1.22	1.53	1	R
	B28 0.4A	1.05	0.71	0.0	0.67	0.00	4.5	1.46	1.03	0.75	1.16	1.46	1	R
	B28 0.4B	1.04	0.71	0.0	0.67	0.00	3.8	1.45	1.02	0.75	1.13	1.45	1	R
	B28 0.4C	1.06	0.68	0.0	0.59	0.00	3.9	1.45	1.07	0.77	1.17	1.45	1	R
	B28 0.4D	0.94	0.66	0.0	0.69	0.00	4.9	1.33	0.95	0.67	1.08	1.33	1	R
	B28 0.4E	0.98	0.66	0.0	0.66	0.00	4.9	1.37	0.99	0.71	1.14	1.37	1	R
	B28 0.4F	0.73	0.69	0.0	1.30	0.00	6.2	1.15	0.73	0.46	0.90	1.15	1	

Invest-igator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd\sqrt{f'_c}}$	$\frac{T_{exp.}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Critical Mode	Restrict-ions
Lyalin	B8 K	0.69	0.0	0.0	0.50	0.00	9.4	0.69	0.69	0.69	1.01	1.01	VEF V	R
	B8 KA	0.84	0.0	0.0	0.50	0.00	7.6	0.84	0.84	0.84	0.98	0.98	VEF	R
	B8 0.1	0.29	1.04	0.44	3.08	0.00	5.8	1.11	0.57	0.08	0.81	1.11	1	
	B8 0.1A	0.30	1.03	0.47	2.86	0.00	6.2	1.11	0.58	0.08	0.86	1.11	1	
	B8 0.2	0.51	0.89	0.42	1.73	0.0	8.1	1.12	0.75	0.27	1.13	1.13	VEF V	
	B8 0.2A	0.54	0.96	0.41	1.78	0.00	7.3	1.20	0.80	0.28	1.09	1.20	1	
	B8 0.4	0.70	0.63	0.30	1.09	0.00	8.8	1.09	0.88	0.53	1.24	1.24	VEF V	
	B8 0.4A	0.75	0.64	0.27	1.05	0.00	8.0	1.13	0.93	0.57	1.16	1.16	VEF	
	B7 0.2	0.62	0.88	0.49	1.13	0.00	7.1	1.20	0.92	0.33	1.31	1.31	VEF	
	B7 0.2A	0.61	0.87	0.45	1.16	0.00	6.5	1.19	0.90	0.32	1.22	1.22	VEF	
	B10 0.2	0.51	1.06	0.40	2.45	0.00	6.8	1.26	0.75	0.27	1.07	1.26	1	
	B10 0.2A	0.50	1.03	0.37	2.42	0.00	6.6	1.23	0.73	0.26	1.00	1.23	1	
	B1	0.53	1.13	0.41	2.04	0.00	5.7	1.33	0.80	0.25	1.03	1.33	1	
	B1A	0.52	1.08	0.48	1.91	0.00	6.4	1.29	0.84	0.21	1.11	1.29	1	
	B2	0.55	1.06	0.39	1.98	0.00	5.7	1.29	0.83	0.26	0.99	1.29	1	
	B2A	0.53	0.98	0.39	1.83	0.00	6.0	1.22	0.81	0.25	0.99	1.22	1	
	B3	0.74	0.70	0.41	1.22	0.00	7.5	1.17	1.03	0.47	1.25	1.25	VEF	
	B3A	0.74	0.70	0.41	1.23	0.00	7.5	1.17	1.03	0.47	1.24	1.24	VEF	
	B5	0.59	0.95	0.40	1.60	0.00	6.4	1.22	0.88	0.28	1.02	1.22	1	
	B5A	0.60	1.03	0.40	1.76	0.00	5.6	1.31	0.91	0.29	1.01	1.31	1	
	B6	0.71	0.93	0.41	1.23	0.00	7.4	1.31	1.00	0.42	1.21	1.31	1	
	B6A	0.77	1.02	0.44	1.24	0.00	7.4	1.43	1.09	0.46	1.32	1.43	1	
	B7	0.85	0.0	0.0	1.35	0.00	12.4	0.85	0.85	0.85	1.32	1.32	VEF V	
	B7A	0.80	0.0	0.0	1.27	0.00	13.5	0.80	0.80	0.80	1.35	1.35	VEF V	
	B8	0.86	0.32	0.12	0.84	0.00	8.2	1.03	0.93	0.78	1.15	1.15	VEF V	R
	B8A	0.90	0.35	0.11	0.89	0.00	7.7	1.09	0.99	0.82	1.07	1.09	1	R
	B9	0.44	0.65	0.86	1.14	0.0	12.4	0.87	0.80	0.22	1.77	1.77	VEF V	
	B9A	0.50	0.66	0.95	0.98	0.00	12.1	0.93	0.89	0.27	2.01	2.01	VEF V	
	B10	0.34	0.58	0.65	1.49	0.00	10.9	0.74	0.64	0.16	1.25	1.25	VEF V	
	B10A	0.48	0.67	0.87	1.11	0.00	12.4	0.91	0.84	0.25	1.84	1.84	VEF V	
	B11	0.47	0.82	0.86	1.44	0.00	12.0	1.04	0.89	0.22	1.70	1.70	VEF V	
	B11A	0.50	0.91	0.83	1.59	0.00	11.7	1.13	0.93	0.22	1.65	1.65	VEF V	
	B12	0.32	0.32	0.35	1.03	0.00	12.9	0.52	0.49	0.20	0.95	0.95	VEF V	
	B12A	0.42	0.43	0.46	1.06	0.00	12.7	0.68	0.63	0.25	1.22	1.22	VEF V	

APPENDIX D. ANALYSIS OF TEST DATA

Investigator	Beam No.	$\frac{T}{T_o}$	$\frac{M}{M_u}$	$\frac{V}{V_o}$	$\frac{r}{r_o}$	$\frac{pf_y}{f'_c}$	$\frac{V_{eff}}{bd/f'_c}$	$\frac{T_{exp}}{T_{theor.}}$ 1	2	3	VEF	Critical $\frac{T_{exp}}{T_{th.}}$	Mode	Restrictions
Yudin	1	0.59	0.69	0.37	0.85	0.28	7.0	1.03	1.08	0.50	1.02	1.08	2	R
	2	0.59	0.69	0.37	0.85	0.28	7.0	1.03	1.08	0.50	1.02	1.08	2	R
	6	0.37	0.87	0.46	1.43	0.28	6.4	1.01	0.89	0.15	0.88	1.01	1	
	7	0.81	0.94	0.50	0.85	0.28	7.0	1.41	1.48	0.69	1.40	1.48	2	R
	10	0.48	1.13	0.60	1.44	0.28	6.4	1.31	1.15	0.19	1.13	1.31	1	
	3	0.68	1.14	0.55	1.69	0.28	8.5	1.46	1.25	0.59	1.55	1.55	VEF V	
	4	0.68	1.14	0.55	1.69	0.28	8.5	1.46	1.25	0.59	1.55	1.55	VEF V	
	8	0.75	1.25	0.61	1.69	0.28	8.5	1.60	1.38	0.65	1.71	1.71	VEF V	
	9	0.75	1.25	0.61	1.69	0.28	8.5	1.60	1.38	0.65	1.71	1.71	VEF V	
	11	0.91	1.16	0.53	1.00	0.00	7.1	1.66	1.36	0.50	1.48	1.66	1	
	12	0.71	0.90	0.41	1.00	0.00	7.1	1.29	1.05	0.39	1.15	1.29	1	
	13	0.71	0.90	0.41	1.00	0.00	7.1	1.29	1.05	0.39	1.15	1.29	1	
	17	0.66	0.59	0.42	0.50	0.20	5.2	1.02	1.13	0.52	1.17	1.17	VEF	R
	18	0.72	0.65	0.46	0.50	0.20	5.2	1.12	1.24	0.57	1.29	1.29	VEF	R
	19	0.72	0.65	0.46	0.50	0.20	5.2	1.12	1.24	0.57	1.29	1.29	VEF	R
	20	0.83	0.75	0.53	0.50	0.20	5.2	1.29	1.43	0.65	1.49	1.49	VEF	R
	21	0.72	0.65	0.46	0.50	0.20	5.2	1.12	1.24	0.57	1.29	1.29	VEF	R
	22	0.70	0.89	0.38	1.00	0.19	4.2	1.28	0.88	0.32	1.05	1.28	1	
Evans and Sarkar	HB1	1.00	0.0	0.0	0.92	0.01	2.9	1.00	1.06	1.15	0.81	1.15	3	
	HB2	0.77	0.78	0.0	1.89	0.01	2.1	1.25	0.82	0.51	0.63	1.25	1	
	HB3	0.48	0.89	0.0	2.68	0.01	1.5	1.10	0.50	0.21	0.40	1.10	1	
	HB4	0.35	0.94	0.0	3.47	0.01	1.1	1.05	0.37	0.12	0.28	1.05	1	
	HB5	0.31	0.97	0.0	3.88	0.01	1.1	1.06	0.33	0.09	0.26	1.06	1	
	HB7	0.81	0.0	0.0	0.87	0.00	3.2	0.81	0.82	0.86	0.65	0.86	3	R
	HB8	0.48	0.83	0.0	2.48	0.00	1.5	1.05	0.49	0.22	0.38	1.05	1	
	HB9	0.38	0.83	0.0	3.00	0.02	1.6	0.98	0.39	0.15	0.32	0.98	1	
	HB10	0.35	0.87	0.0	3.28	0.01	1.2	0.99	0.36	0.13	0.27	0.99	1	
	HB11	0.29	0.91	0.0	3.91	0.01	1.1	0.99	0.30	0.08	0.24	0.99	1	
	HB13	0.94	0.0	0.0	0.87	0.00	2.9	0.94	0.93	0.99	0.67	0.99	3	R
	HB14	0.69	0.57	0.0	1.70	0.01	2.2	1.03	0.70	0.50	0.50	1.03	1	
	HB15	0.55	0.84	0.0	2.31	0.00	1.5	1.11	0.54	0.27	0.39	1.11	1	
	HB16	0.39	0.90	0.0	3.11	0.01	1.4	1.05	0.40	0.15	0.30	1.05	1	
	HB17	0.32	0.96	0.0	3.75	0.01	0.9	1.06	0.33	0.10	0.23	1.06	1	