Truncation error treatment for the model-free implied moment estimator

Author:
Lee, Geul

Publication Date:
2015

DOI:
https://doi.org/10.26190/unsworks/18851

Truncation error treatment
for the model-free implied moment estimator

Geul Lee

A thesis in fulfillment of the requirements for the degree of

Doctor of Philosophy

School of Banking and Finance

UNSW Business School

August 2015

PLEASE TYPE

THE UNIVERSITY OF NEW SOUTH WALES
Thesis/Dissertation Sheet

Surname or Family name: **Lee**

First name: **Geul** Other name/s:

Abbreviation for degree as given in the University calendar: **PhD**

School: **School of Banking and Finance** Faculty: **UNSW Business School**

Title: **Truncation error treatment for the model-free implied moment estimator**

Abstract 350 words maximum: (PLEASE TYPE)

This thesis investigates the impact of truncation, that is, the complete unavailability of significantly deep-out-of-the-money option price quotes, on the implied moment estimators of Bakshi et al. (2003) and suggests a new truncation treatment method that makes truncation error, or estimation bias due to truncation, less volatile. Although previous studies have already suggested two truncation error reduction methods for model-free implied moment estimation, these methods may not be able to effectively reduce truncation error when they are used with the implied skewness or kurtosis estimators, which rely more heavily on deep-out-of-the-money option prices. Hence, we first test whether the two existing methods, specifically, the linear extrapolation method of Jiang and Tian (2005) and the domain symmetrisation method of Dennis and Mayhew (2002), can reduce truncation error effectively even when they are used in conjunction with the two higher moment estimators. The test results show that the truncation error reduction effect may be incomplete for both methods when they are used for implied skewness or kurtosis estimation. Given this result, we further investigate the relationship between truncation level and truncation error size, and then propose an alternative method of truncation error treatment, namely, domain stabilisation, based on the relationship identified. The tests on the effectiveness of domain stabilisation reveal that although this method increases the mean size of the truncation error, it also makes the size less volatile across different observations. This result implies that when our new method is employed, truncation has less impact on cross-sectional comparison and on tracking the time-series dynamics of implied moments.

Originality Statement

'I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.'

Signed GEUL LEE

Date 31 August 2015

Copyright Statement

GEUL LEE

Authenticity Statement

‘I certify that the Library deposit digital copy is a direct equivalent of the final officially approved version of my thesis. No emendation of content has occurred and if there are any minor variations in formatting, they are the result of the conversion to digital format.’

GEUL LEE

Abstract

This thesis investigates the impact of truncation, that is, the complete unavailability of significantly deep-out-of-the-money option price quotes, on the implied moment estimators of Bakshi et al. (2003) and suggests a new truncation treatment method that makes truncation error, or estimation bias due to truncation, less volatile. Although previous studies have already suggested two truncation error reduction methods for model-free implied moment estimation, these methods may not be able to effectively reduce truncation error when they are used with the implied skewness or kurtosis estimators, which rely more heavily on deep-out-of-the-money option prices. Hence, we first test whether the two existing methods, specifically, the linear extrapolation method of Jiang and Tian (2005) and the domain symmetrisation method of Dennis and Mayhew (2002), can reduce truncation error effectively even when they are used in conjunction with the two higher moment estimators. The test results show that the truncation error reduction effect may be incomplete for both methods when they are used for implied skewness or kurtosis estimation so that the estimate can be significantly different from the true value. Given this result, we further investigate the relationship between truncation level and truncation error size, and then propose an alternative method of truncation error treatment, namely, domain stabilisation, based on the relationship identified. The tests on the effectiveness of domain stabilisation reveal that although this method increases the mean size of the truncation error, it also makes the size less volatile across different observations. This result implies that when our new method is employed, truncation has less impact on cross-sectional comparison of implied moments among the options on different underlying assets and on tracking the time-series dynamics of implied moments.

Acknowledgements

Some of the work presented in this thesis has been circulated in the form of working papers as follows:

- Lee, G., and Yang, L., 'Impact of truncation on model-free implied moment estimator' (http://papers.ssrn.com/abstract=2485513, Chapters 2, 4, and 5); and
- Lee, G. 'Effectiveness of linear extrapolation in model-free implied moment estimation' (http://papers.ssrn.com/abstract=2590040, Chapter 3).

In addition, the above-mentioned papers have been presented at the following academic conferences:

- Lee, G., and Yang, L. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Auckland Finance Meeting, Auckland, New Zealand;
- Lee, G. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Auckland Finance Meeting, Auckland, New Zealand;
- Lee, G., and Yang, L. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Conference on Asia-Pacific Financial Markets, Seoul, Republic of Korea;
- Lee, G., and Yang, L. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Financial Management Association Annual Meeting, Orlando, United States;
- Lee, G. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Asia-Pacific Association of Derivatives Conference, Busan, Republic of Korea;
- Lee, G., and Yang, L. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Derivatives Markets Conference, Auckland, New Zealand;
- Lee, G. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Derivatives Markets Conference, Auckland, New Zealand;

- Lee, G., and Yang, L. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Asian Finance Association Annual Meeting, Changsha, China;

- Lee, G., and Yang, L. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', International Finance and Banking Society Conference, Hangzhou, China;

- Lee, G. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Internation Conference of the Financial Engineering and Banking Society, Nantes, France;

- Lee, G. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Vietnam International Conference in Finance, Ho Chi Minh City, Vietnam;

- Lee, G., and Yang, L. (2015), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Financial Markets and Corporate Governance Conference, Fremantle, Australia; and

- Lee, G., and Yang, L. (2014), 'Effectiveness of linear extrapolation in model-free implied moment estimation', Australasian Finance and Banking Conference, Sydney, Australia.

For the part of the work in this thesis that is also included in the paper 'Impact of truncation on model-free implied moment estimator', the greater part of work is directly attributable to me as a doctoral student. Associate Professor Li Yang, who is my supervisor and the co-author of the paper, contributed to the research by discussions on concepts, ideas, and research directions, and also participated in the editorial process. However, the major part of the research was conducted solely by me in keeping with the requirements of the University of New South Wales Conditions for Award of Doctor of Philosophy Policy.

Contents

the execution of

List of Tables

List of Figures

List of Abbreviations

APC: absolute percentage change

BS: Black-Scholes constant volatility

DITM: deep-in-the-money

DOTM: deep-out-of-the-money

DStab: domain stabilisation

DSym: domain symmetrisation

IVAL: implied volatility adjusted log-moneyness

JBF: Journal of Banking and Finance

JF: Journal of Finance

JFE: Journal of Financial Economics

JFM: Journal of Futures Markets

LE: linear extrapolation

NTM: near-the-money

OTM: out-of-the-money

RF: Review of Finance

RFS: Review of Financial Studies

RND: risk-neutral density

SVJ: stochastic volatility and jump

Chapter 1

Introduction

Option prices imply the probability density function of the underlying asset price at maturity under the risk-neutral probability measure. Breeden and Litzenberger (1978) prove that an Arrow-Debreu security, that is, a security that pays one dollar if a certain state occurs at a given date and zero otherwise, can be synthesised using options and that the risk-neutral density (RND) can be derived from the prices of the synthesised Arrow-Debreu securities. Following this discovery, a stream of papers suggest how RND can be retrieved from the option prices (e.g., Shimko, 1993; Jackwerth and Rubinstein, 1996; Malz, 1997; Aït-Sahalia and Lo, 1998; Campa et al., 1998; Bliss and Panigirtzoglou, 2000; Gemmill and Saflekos, 2000). Since the implied RND is the implied probability density of the underlying asset price at maturity under the risk-neutral density, the density is closely linked to the expectations of the option market traders about the underlying price movement between the current time and the maturity date. Hence, the implied RND has been regarded as a valuable source of market information and thus has been adopted by numerous studies as the topic of research.

One drawback of using the implied RND as a source of information, however, is that it is not in the form of a variable but of a function, and therefore, it is difficult to consider the RND itself as a variable during empirical analysis. Given this issue, many studies instead employ the moments of the implied RND as variables to summarise the characteristics of the RND. Since the moments contain information about the shape of the RND and because it is straightforward to interpret the information in each moment, they have been used in several recent studies of the options market.[1]

[1] Similarly, the slope and curvature of the Black-Scholes implied volatility curve have been used by recent studies as well (See, e.g., Doran and Krieger, 2010). As shown by Zhang and Xiang (2008), there is a close relationship between the higher moments of the implied RND and the shape of the implied volatility curve.

An issue that might be of concern when employing the moments of the implied RND for an empirical analysis is that deriving the moments can be a computationally burdensome task, since one first needs to construct the RND from the option prices and then calculate the moments. To overcome this issue, recent studies have tried to derive the moments directly from the option prices without constructing the complete density, either parametrically or non-parametrically (e.g., Corrado and Su, 1997; Britten-Jones and Neuberger, 2000; Bakshi et al., 2003; Backus et al., 2004; Zhang and Xiang, 2008; Tian, 2011). Between the parametric and non-parametric methods, the latter has been more popularly used due to its model-freeness, which renders the method free from the model misspecification error. For instance, the implied volatility estimator of Britten-Jones and Neuberger (2000) and the implied volatility, skewness, and kurtosis estimators of Bakshi et al. (2003) have been employed in several recent studies in order to extract information from option prices. In particular, the implied skewness and kurtosis estimators of Bakshi et al. (2003) are drawing an increasing amount of attention from both academics and practitioners, since such higher moments can be used to forecast the extreme movements of the underlying asset price and, therefore, to manage catastrophic risk. Table 1.1 lists some of the studies that employ the implied higher moment estimators of Bakshi et al. (2003).

Although the model-free implied moment estimators are theoretically more robust due to their model-freeness, they suffer from some empirical issues that stem from the limited availability of option prices. In order to estimate implied moments non-parametrically, one needs an infinite set of options with strike prices that span a continuum over the positive real line. However, as noted by Jiang and Tian (2005), it is virtually impossible to obtain this continuum of options for two reasons. First, market participants are only allowed to quote option prices for a discrete set of strike prices. Second, it is completely impossible to observe reliable option quotes for a large part of the deep-out-of-the-money (DOTM) or, equivalently, deep-in-the-money (DITM) region of the strike price domain, and, furthermore, some of the DOTM option quotes that are available need to be filtered for research use because of liquidity and market-microstructure issues. For instance, the relative size of a price tick to an option price becomes larger when the option becomes cheaper, and therefore, extremely low option price quotes are regarded as unreliable and so discarded.[2]

[2] The restrictiveness of the quote-based filter is rather arbitrary and there is no consensus in the literature on the threshold price level beyond which the observations should be excluded. For instance, the threshold

Between the two issues, it is relatively easy to deal with the issue of strike price discreteness, because there exist various interpolation techniques that enable the approximation of option prices for a continuum of strike prices using the option prices for a discrete set of strike prices. This can be acheived in a reliable manner since the interpolation can be conducted on the Black-Scholes implied volatility curve, not on the option prices themselves.

Table 1.1: List of studies employing the implied moment estimators of Bakshi et al. (2003)

This table presents a non-exhaustive list of papers in which Bakshi et al.'s (2003) model-free implied moment estimators are employed. The first column shows the name of the author(s), the year the paper is published, and the name of the journal in which the paper is published. In the first column, the following abbreviations are used: JBF (Journal of Banking and Finance), JF (Journal of Finance), JFE (Journal of Financial Economics), JFM (Journal of Futures Markets), RF (Review of Finance), and RFS (Review of Financial Studies). The second column lists the moment(s) estimated in the corresponding paper. The third column shows which truncation error reduction method is employed in the corresponding paper. In the third column, the following abbreviations are used: DSym (domain symmetrisation method of Dennis and Mayhew (2002)), and LE (linear extrapolation method of Jiang and Tian (2005)). The dashes in the third column indicate that we cannot find any description about the truncation reduction method for the corresponding paper; however, this does not necessarily mean that the corresponding paper does not consider the truncation error.

Paper	Moments estimated	Truncation error reduction method
Dennis and Mayhew (2002, JFQA)[3]	skewness	DSym
Han (2008, RFS)	skewness	-
Lin et al. (2008, JFM)	skewness and kurtosis	-
Duan and Wei (2009, RFS)	skewness and kurtosis	etc.
Xing et al. (2010, JFQA)	skewness	-
Chou et al. (2011, JFM)	volatility, skewness, and kurtosis	-
Buss and Vilkov (2012, RFS)	volatility and skewness	LE
Chang et al. (2012, RF)	variance and skewness	LE
Diavatopoulos et al. (2012, JBF)	skewness and kurtosis	-
Bali and Murray (2013, JFQA)	skewness	etc.
Chang et al. (2013, JFE)	volatility, skewness, and kurtosis	LE
Conrad et al. (2013, JF)	volatility, skewness, and kurtosis	DSym
DeMiguel et al. (2013, JFQA)	volatility and skewness	LE
Kozhan et al. (2013, RFS)	kurtosis	LE
Neumann and Skiadopoulos (2013, JFQA)	volatility, skewness, and kurtosis	LE

For instance, Jiang and Tian (2005) employ the cubic spline function, while Broadie et al.

mid-point price is set as 0.125 in Han (2008) and Xing et al. (2010), 0.375 in Bakshi et al. (1997) and Jiang and Tian (2005), and 0.5 in Conrad et al. (2013). At an extreme, CBOE uses every quote for the options on the S&P 500 index to calculate the CBOE volatility index unless the bid quote is zero, and this greatly reduces the issue of information loss at the cost of more liquidity and microstructure-related issues. Chapter 3 implies that neither one has significant impact on the CBOE volatility index because DOTM options have little effects on implied volatility estimation.

(2007) adopt the piecewise quadratic function to generate the continuous Black-Scholes implied volatility curve. Since the severe convexity of the option price function with respect to the strike price can be alleviated considerably by converting the option prices to Black-Scholes implied volatility levels, interpolation can be performed more effectively via the implied volatility function.

In contrast, it is relatively difficult to approximate option prices for the DOTM-DITM region of the strike price domain for which option prices are completely unavailable. While there are at least two reference points for the strike prices for which interpolation can be performed, there is at most only one reference point beyond the minimum and maximum strike prices for which the option price is available. Hence, a strong assumption needs to be made to approximate option prices for the region of the strike price domain for which option prices are completely unavailable. This difficult task should still be accomplished, however, because omitting this extrapolation procedure may induce large estimation errors, especially for the implied skewness and kurtosis estimators of Bakshi et al. (2003), which are more closely related to the tail shape of the implied RND. Jiang and Tian (2005) refer to these errors as 'truncation errors' because the error is induced by the unavailable DOTM option prices that look as if they are 'truncated'.

So far, two methods have been proposed by previous studies to reduce the size of the truncation error, that is, the estimation bias due to truncation. First, Jiang and Tian (2005) suggest linear extrapolation (LE), which generates DOTM option prices by extending the Black-Scholes volatility curve with flat lines that can then be converted to option prices. Second, Dennis and Mayhew (2002) suggest domain symmetrisation (DSym), which additionally filters the available option prices until the minimum and maximum strike prices become equidistant from the underlying asset price. As Table 1.1 demonstrates, both methods have been adopted in recent studies, although the former has been more popular. One reason for this popularity is that the linear extrapolation method can be applied to any implied moment estimator, whereas the domain symmetrisation method is suggested for only the implied skewness estimator of Bakshi et al. (2003).

Although both truncation error reduction methods successfully point out the drawbacks in the model-free implied moment estimation and so are potentially effective in alleviating the truncation error, several questions can still be addressed toward the implementation of these methods, especially when the implied skewness and kurtosis estimators of Bakshi et al. (2003) are considered. First, is LE, which was originally suggested for the implied volatility estimator of Britten-Jones and Neuberger (2000), also effective for the implied skewness and kurtosis estimators? Jiang and Tian (2005) derive the upper bound of the truncation error for the implied volatility estimator of Britten-Jones and Neuberger (2000) and show that the truncation error is negligible if the endpoints of the integration domain are more than two standard deviations away from the forward price. However, when skewness or kurtosis of the implied RND is estimated, both the upper bounds and the negligibility of the truncation error need to be re-examined. The upper bound of the truncation error can be less tight in this case, i.e., the unavailability of DOTM option prices may induce a larger truncation error, because the higher moments are more closely related to the tails of the density whose curvature is implied by DOTM option prices. Second, does DSym remain effective even when the implied RND is asymmetric? Dennis and Mayhew (2002) show that DSym can reduce the truncation error of the implied skewness estimator of Bakshi et al. (2003) while assuming that the underlying price follows a constant volatility process with no jumps, with which the implied RND is symmetric. Several empirical studies, however, show that the implied RND is severely asymmetric in many major options markets.

To answer these questions, this thesis investigates the impact of truncation on the implied moment estimators of Bakshi et al. (2003). Specifically, we investigate how the impact of truncation on the implied skewness and kurtosis estimators differs from the impact on the implied volatility estimator, how the truncation level and the truncation error size should be defined and measured, and how the relationship between the truncation level and the truncation error size can be explained. In addition, we investigate the effectiveness of LE and DSym as truncation error reduction methods, using both generated and observed option prices. Finally, we propose an alternative method of truncation error treatment, which is called domain stabilisation (DStab), that reduces the volatility of the truncation error either cross-sectionally or over time. Specifically, we show how the truncation error can be made less volatile by properly measuring and controlling the level of truncation.

Chapter 2 provides an overview of the model-free implied moment estimators of Bakshi et al. (2003) and conducts a theoretical analysis on the impact of truncation on the estimators. We first explain how the moments of the implied RND can be directly calculated from OTM option prices without constructing the RND itself, and then describe the framework of the implied moment estimators. Next, we demonstrate how the implied moment estimates are affected by truncation using two sets of model-based generated option prices.

Chapter 3 shows how the effectiveness of LE can be measured for a finite set of option prices, and then evaluates the efficiency of LE for the S&P 500 index options data. Since the availability of market option prices is limited in almost every options market, it is impossible to derive the true level of the implied moment using the option prices that are observed from markets, and therefore, we cannot measure the size of truncation error with these observed prices. Given this issue, most related studies demonstrate and measure the truncation error using model-based generated option prices (e.g., Dennis and Mayhew, 2002, 2009; Jiang and Tian, 2005). Chapter 3 circumvents this problem by using a different approach to evaluate the effectiveness of LE. We show that the sensitivity of the implied moment estimates to a marginal change in option price availability can be formulated, regardless of whether LE is applied or not. Using the S&P 500 index options data and the sensitivity formulae, Chapter 3 investigates how effectively LE makes the implied moment estimators less sensitive to option price unavailability. If the size of the truncation error is effectively reduced so that the impact of truncation on the moment estimate is negligible, the estimate must not change significantly even when there is a change in the availability of option prices. Namely, if the moment estimate varies abruptly when the estimation is re-executed following a small change in the option prices availability while LE is applied consistently, then it can be conjectured that the truncation error has not been fully alleviated by LE. Hence, the sensitivity of the moment estimate to the change in option price availability can be used to measure the effectiveness of LE. The empirical results in Chapter 3 suggest that although the size of the truncation error is reduced significantly by the application of LE regardless of the moment estimated, the remaining error can be too large to be regarded as negligible for the implied skewness and kurtosis estimators.

Chapter 4 analyses the effectiveness of DSym. Although the test results of Dennis and Mayhew (2002) provide a deep insight into the implied skewness estimator, there is still a need to further examine the effectiveness of DSym given that the implied skewness estimator is popularly used and that DSym has been adopted by recent studies (e.g., Conrad et al., 2013). We explain the relationship between option price unavailability and estimation bias, and then examine the impact of domain asymmetry on the implied skewness estimator. Specifically, we investigate how the integration domain symmetry should be defined and the asymmetry level should be measured, and then examine whether DSym effectively reduces the truncation error of the implied skewness estimator even when the implied RND is asymmetric. This chapter shows that DSym is effective if the domain symmetry is defined in terms of log-moneyness and the implied RND is symmetric, but that the truncation error may not be reduced by DSym if either of those two conditions is violated.

Chapter 5 introduces DStab, which makes the size of the truncation error less volatile either cross-sectionally or over time. The issues regarding of LE and DSym imply that it is very difficult to eliminate the truncation error, especially for the implied skewness and kurtosis estimators, and therefore there is a need to consider the issue from a different perspective. If it is difficult to eliminate the truncation error, then is there any other way to control the impact of truncation on model-free implied moment estimation? In order to find such a way out, one first needs to clarify the objective of implied moment estimation. Is the estimation conducted to identify the true level of a moment? If not, is it done to track the dynamics of the implied moments over time, or to make a comparison of the implied moment levels among options on different underlying assets? If implied moments are estimated in order to capture the dynamics or the relative level of implied moments, stabilising the truncation error over time could be an alternative of minimising the error. In other words, if the level of truncation error can be maintained at a fixed level cross-sectionally and over time, the de facto effect of truncation on implied moment estimation will be minimised.

Chapter 4 suggests that it is not the strike price but the log-moneyness of the endpoints of the integration domain that should be considered when interpreting the impact of truncation on the model-free implied moment estimator. In addition, Chapter 5 shows that the level of implied volatility also needs to be considered in order to explain the relationship between the truncation level and the truncation error size. Based on these findings, Chapter 5 reveals that if the level of truncation is defined using the endpoint log-moneyness that is adjusted by the level of implied volatility, a strong relationship between the truncation level and the truncation error size can be found. An empirical analysis of

S&P 500 index options suggests that the variance of the truncation error decreases when DStab is employed, whereas both the mean and variance increase when DSym is utilised instead.

Overall, this thesis not only suggests a new way to reduce the impact of truncation on model-free implied moment estimation, but also provides a deeper understanding of the relationship between the truncation level and the size of the truncation error. We show that one must consider the log-moneyness of endpoint strike prices and the level of implied volatility when measuring the degree of truncation. By analysing the relationship between the truncation level and the truncation error size, and by introducing a new way to control truncation, this thesis provides a methodological foundation for further studies on the higher moments of the implied RND, and so makes the model-free implied moment estimators of Bakshi et al. (2003) more reliable to employ.

Chapter 2

Theoretical analysis on the model-free implied moment estimator and truncation

Chapter Summary

This chapter provides an overview of the model-free implied moment estimators of Bakshi et al. (2003) and conducts a theoretical analysis on the impact of truncation on the estimators. First, we explain how the moments of the implied RND can be directly calculated from OTM option prices without constructing the RND itself. Next, we describe the framework of the implied moment estimators. Finally, we show how truncation can affect the implied skewness and kurtosis estimators using two sets of model-based generated option prices.

2.1. Introduction

The most impressive characteristic of the implied moment estimators of Bakshi et al. (2003) is that they can derive the moments of the implied RND directly from the option prices without constructing the RND nor relying on any assumptions on the underlying price dynamics. Although this feature gives the estimators both theoretical robustness and implementational convenience, it also makes it more difficult to find out the logic behind the estimators and understand how the estimators work. In order to estimate the moments of the implied RND without constructing the RND itself, we need a technique that enables the estimators to circumvent the necessity of the RND. The problem is that this kind of technique is inevitably less intuitive than the direct calculation of the moments from the RND.

Understanding the logic behind the implied moment estimators becomes even more necessary when an assumption made by the estimators is violated. As mentioned in Chapter 1, although the estimators implicitly assume that OTM option prices are observable for the continuum of strike price from zero to positive infinity, it is virtually impossible to obtain all these option prices in almost every options market. However, given that the option prices are not used to construct the implied RND but directly passed to the estimators as inputs, it is difficult to determine how the unavilability of option prices affects the estimation procedure, and therefore, we must well understand the methodology behind the estimators to evaluate the impact of truncation on the estimators.

Hence, this chapter provides an overview of the model-free implied moment estimators of Bakshi et al. (2003) and conducts a theoretical analysis of the impact of truncation on implied moment estimation for a better understanding of the estimators. We first describe the concept of payoff spanning which is a fundamental building block of the estimators, and then explain how the moments of the implied RND can be derived by payoff spanning. Next, we conduct a theoretical analysis of the impact of truncation on implied moment estimators. Two sets of model-based generated option prices are used to empirically show how the implied moment estimators are affected by truncation.

The rest of this chapter is organised as follows. Section 2.2 elaborates how the moments of the implied RND can be derived by the estimators of Bakshi et al. (2003), and explains how truncation should be interpreted when the implied moment estimators are used.

Section 2.3 describes the S&P 500 index options data used in this chapter and throughout this thesis. Section 2.4 examines the impact of truncation on the implied moment estimates using model-based generated option prices. Section 2.5 concludes this chapter.

2.2. Model-free implied moment estimators

This section provides an overview of the model-free implied moment estimators of Bakshi et al. (2003). Section 2.2.1 describes how payoff functions can be spanned using OTM option prices. Section 2.2.2 shows how the implied moment estimators can be constructed by payoff spanning. Section 2.2.3 explains how truncation should be interpreted when the implied moment estimators are used.

2.2.1 Payoff spanning

Carr and Madan (2001) show that any twice-continuously differentiable payoff functions $H[S] \in \mathcal{C}^2$ of the underlying price S can be expanded as

$$\begin{aligned} H[S] &= H[\bar{S}] + (S-\bar{S})H_S[\bar{S}] + \mathbb{I}_{S>\bar{S}} \int_{\bar{S}}^{\infty} H_{SS}[K](S-K)dK \\ &\quad + \mathbb{I}_{S<\bar{S}} \int_{0}^{\bar{S}} H_{SS}[K](K-S)dK \\ &= H[\bar{S}] + (S-\bar{S})H_S[\bar{S}] + \int_{\bar{S}}^{\infty} H_{SS}[K](S-K)^{+}dK \\ &\quad + \int_{0}^{\bar{S}} H_{SS}[K](K-S)^{+}dK, \end{aligned} \tag{2.1}$$

where $\mathbb{I}_C$ is an indicator function whose value is one when the condition C holds and zero otherwise, $H_S[\cdot]$ and $H_{SS}[\cdot]$ represent the first and second derivatives of the payoff function with respect to S, and $\bar{S}$ is a nonnegative real constant. In addition, Carr and Madan (2001) also show that the arbitrage-free price of $H[S]$, i.e., the discounted expected value of $H[S]$ under the risk-neutral probability measure, can be derived as

$$\begin{aligned} \mathbb{E}_t^*\{e^{-r\tau}H[S]\} &= (H[\bar{S}] - \bar{S}H_S[\bar{S}])e^{-r\tau} + H_S[\bar{S}]S(t) + \int_{\bar{S}}^{\infty} H_{SS}[K]C(t,\tau;K)dK \\ &\quad + \int_{0}^{\bar{S}} H_{SS}[K]P(t,\tau;K)dK, \end{aligned} \tag{2.2}$$

where $\mathbb{E}_t^*\{\cdot\}$ is the expectation operator under the risk-neutral density with respect to the filtration $\mathcal{F}$ for time t, $C(t,\tau;K)$ and $P(t,\tau;K)$ denote the OTM call and put prices at time t for time to maturity τ and strike price K, respectively.

2.2.2 Construction of implied moment estimators

The volatility, skewness, and kurtosis of a random variable X are defined as

$$\mathrm{VOL}(X) = \left[\mathbb{E}\left\{(X-\mathbb{E}[X])^2\right\}\right]^{1/2}, \tag{2.3}$$

$$\mathrm{SKEW}(X) = \frac{\mathbb{E}\left\{(X-\mathbb{E}[X])^3\right\}}{\left[\mathbb{E}\left\{(X-\mathbb{E}\left[X\right])^2\right\}\right]^{3/2}}, \tag{2.4}$$

$$\mathrm{KURT}(X) = \frac{\mathbb{E}\left[(X-\mathbb{E}[X])^4\right\}}{\left[\mathbb{E}\left\{(X-\mathbb{E}\left[X\right])^2\right\}\right]^{2}}, \tag{2.5}$$

respectively. If we expand Equations (2.3)–(2.5), we obtain

$$\mathrm{VOL}(X) = \left\{\mathbb{E}[X^2]-\mathbb{E}[X]^2\right\}^{1/2}, \tag{2.6}$$

$$\mathrm{SKEW}(X) = \frac{\mathbb{E}[X^3]-3\mathbb{E}[X^2]\mathbb{E}[X]+2\mathbb{E}[X]^3}{\left\{\mathbb{E}\left[X^2\right]-\mathbb{E}\left[X\right]^2\right\}^{3/2}}, \tag{2.7}$$

$$\mathrm{KURT}(X) = \frac{\mathbb{E}[X^4]-4\mathbb{E}[X^3]\mathbb{E}[X]+6\mathbb{E}[X^2]\mathbb{E}[X]^2-3\mathbb{E}[X]^4}{\left\{\mathbb{E}\left[X^2\right]-\mathbb{E}\left[X\right]^2\right\}^{2}}, \tag{2.8}$$

respectively. Equations (2.6)–(2.8) imply that we can calculate the volatility, skewness, and kurtosis of a random variable X if we know the expected value of X, X^2, X^3, and X^4 under the corresponding probability measure.

The random variable and the probability measure for which Bakshi et al. (2003) target to obtain the moments of the probability density is the log-return of the underlying asset S between time t and $t+\tau$, i.e., $\ln\left[S(t+\tau)/S(t)\right]$, and the risk-neutral probability measure $\mathbb{P}^*$, respectively. If we denote the log-return as $R(t,\tau)$ as in Bakshi et al. (2003), we need to know the expected values $\mathbb{E}^*[R(t,\tau)]$, $\mathbb{E}^*[R(t,\tau)^2]$, $\mathbb{E}^*[R(t,\tau)^3]$, and $\mathbb{E}^*[R(t,\tau)^4]$ under $\mathbb{P}^*$. In order to obtain the expected values, Bakshi et al. (2003) introduce payoff spanning to obtain the fair value at time t of the volatility contract V, cubic contract W, and quartic contract X whose payoffs at time $t+\tau$ are $R(t,\tau)^2$, $R(t,\tau)^3$, and $R(t,\tau)^4$, respectively. Since the arbitrage-free price of V, W, and X are equal to $\mathbb{E}^*[e^{-r\tau}R(t,\tau)^2]$, $\mathbb{E}^*[e^{-r\tau}R(t,\tau)^3]$, and $\mathbb{E}^*[e^{-r\tau}R(t,\tau)^4]$, respectively, we can obtain the last three of the required expected values by multiplying the fair values by $e^{r\tau}$.

To obtain the fair values, Bakshi et al. (2003) set the second derivative H_{SS} in Equation (2.2) for V, W, and X as

$$H_{SS}^{V}[K] = \frac{2\left(1-\ln\left[\frac{K}{S(t)}\right]\right)}{K^2}, \tag{2.9}$$

$$H_{SS}^{W}[K] = \frac{6\ln\left[\frac{K}{S(t)}\right] - 3\left(\ln\left[\frac{K}{S(t)}\right]\right)^2}{K^2}, \tag{2.10}$$

$$H_{SS}^{X}[K] = \frac{12\left(\ln\left[\frac{K}{S(t)}\right]\right)^2 - 4\left(\ln\left[\frac{K}{S(t)}\right]\right)^3}{K^2}, \tag{2.11}$$

respectively. Given Equations (2.2) and (2.9)–(2.11), the arbitrage-free price of V, W, and X can be defined as

$$\begin{aligned} V(t,\tau) = & \int_{S(t)}^{\infty} \frac{2\left(1-\ln\left[\frac{K}{S(t)}\right]\right)}{K^2} C(t,\tau;K)dK \\ & + \int_{0}^{S(t)} \frac{2\left(1+\ln\left[\frac{S(t)}{K}\right]\right)}{K^2} P(t,\tau;K)dK, \end{aligned} \tag{2.12}$$

$$\begin{aligned} W(t,\tau) = & \int_{S(t)}^{\infty} \frac{6\ln\left[\frac{K}{S(t)}\right] - 3\left(\ln\left[\frac{K}{S(t)}\right]\right)^2}{K^2} C(t,\tau;K)dK \\ & - \int_{0}^{S(t)} \frac{6\ln\left[\frac{S(t)}{K}\right] + 3\left(\ln\left[\frac{S(t)}{K}\right]\right)^2}{K^2} P(t,\tau;K)dK, \end{aligned} \tag{2.13}$$

$$\begin{aligned} X(t,\tau) = & \int_{S(t)}^{\infty} \frac{12\left(\ln\left[\frac{K}{S(t)}\right]\right)^2 - 4\left(\ln\left[\frac{K}{S(t)}\right]\right)^3}{K^2} C(t,\tau;K)dK \\ & + \int_{0}^{S(t)} \frac{12\left(\ln\left[\frac{S(t)}{K}\right]\right)^2 + 4\left(\ln\left[\frac{S(t)}{K}\right]\right)^3}{K^2} P(t,\tau;K)dK, \end{aligned} \tag{2.14}$$

where $C(t,\tau;K)$ and $P(t,\tau;K)$ denote the price of call and put options with strike price K and time to maturity τ at day t, respectively. Next, Bakshi et al. (2003) approximate the value of $\mathbb{E}^*[R(t,\tau)]$ as

$$\mu(t,\tau) = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}V(t,\tau) - \frac{e^{r\tau}}{6}W(t,\tau) - \frac{e^{r\tau}}{24}X(t,\tau). \tag{2.15}$$

Then, given Equations (2.6)–(2.8), the τ-period implied risk-neutral volatility, skewness, and kurtosis can be defined as

$$\mathrm{VOL}(t,\tau) = \left[e^{r\tau}V(t,\tau) - \mu(t,\tau)^2\right]^{1/2}, \tag{2.16}$$

$$\mathrm{SKEW}(t,\tau) = \frac{e^{r\tau}W(t,\tau) - 3\mu(t,\tau)e^{r\tau}V(t,\tau) + 2\mu(t,\tau)^3}{[e^{r\tau}V(t,\tau) - \mu(t,\tau)^2]^{3/2}}, \tag{2.17}$$

$$\mathrm{KURT}(t,\tau) = \frac{e^{r\tau}X(t,\tau) - 4\mu(t,\tau)e^{r\tau}W(t,\tau) + 6e^{r\tau}\mu(t,\tau)^2V(t,\tau) - 3\mu(t,\tau)^4}{[e^{r\tau}V(t,\tau) - \mu(t,\tau)^2]^2}. \tag{2.18}$$

2.2.3 Interpretation of limited option price availability

If OTM option prices are available only for a strike price domain $[K_{\min}(t,\tau), K_{\max}(t,\tau)]$ for time t and maturity τ, where $0 \leq K_{\min}(t,\tau) \leq S(t) \leq K_{\max}(t,\tau) < \infty$, and integrations are conducted only for this domain for the fair value estimation of V, W, and X, then it is equivalent to assuming that the OTM option price is zero for the strike price domains $[0, K_{\min}(t,\tau))$ and $(K_{\max}(t,\tau), \infty)$, i.e., $P(t,\tau;K) \equiv 0$ for $\{K : 0 < K < K_{\min}(t,\tau)\}$ and $C(t,\tau;K) \equiv 0$ for $\{K : K_{\max}(t,\tau) < K < \infty\}$. The following proposition shows how this assumption affects the fair value estimation:

Proposition 2.1. If truncation exists for the strike price domain $(0, K_{\min}(t,\tau))$ and $(K_{\max}(t,\tau), \infty)$, and the fair value of $V, W,$ and X are estimated without considering the OTM option prices on the truncated domain, it is equivalent to assuming that, for the risk-neutral probability measure $\mathbb{P}^*$,

$$\begin{aligned}
&\mathbb{P}^*_t\left\{\ln\left[\frac{S(t+\tau)}{S(t)}\right] < \ln\left[\frac{K_{\min}(t,\tau)}{S(t)}\right]\right\} = 0; \quad \text{and} \\
&\mathbb{P}^*_t\left\{\ln\left[\frac{S(t+\tau)}{S(t)}\right] > \ln\left[\frac{K_{\max}(t,\tau)}{S(t)}\right]\right\} = 0,
\end{aligned} \tag{2.19}$$

where $\mathbb{P}^*_t\{\cdot\}$ is the conditional probability operator for the measure $\mathbb{P}^*$ with respect to the filtration $\mathcal{F}_t$ for time t.

Proof. See Appendix A. □

In this thesis, Proposition 2.1 is assumed to cause a truncation of the implied risk-neutral density, given that some of the unavailable DOTM option prices are in fact observed as nonzero value but discarded during the data filtration process in almost every case. This means that at least for some of the DOTM options for which the option price is regarded as unavailable, market participants think that it is possible for the underlying price to reach the corresponding strike prices at maturity.

2.3. Data

The S&P 500 index options dataset used in this thesis spans the eleven-year sample period from January 2000 to December 2010. The option prices and risk-free rate yield curve data are retrieved from IvyDB OptionMetrics via Wharton Research Data Services. The closing option price is estimated as the mid-point between the closing bid and ask prices. The risk-free rate for each day and maturity is estimated by linearly interpolating the two most adjacent points on the corresponding daily yield curve for which the rate is observable. OptionMetrics also distributes the option-implied dividend rate data, and this rate is used to approximate the dividend rate $q(t, T)$ for day t_0 and maturity date T as

$$q(t_0, T) = \left[\prod_{i=1}^{n} (1 + q^*(t_i)) \right]^{1/n} - 1, \tag{2.20}$$

where n is the total number of implied dividend rate observations available for days between t_0 and T, $q^*(t_i)$ is the implied dividend rate for day t_i, which is between t_0 and T. This dividend rate is used to calculate the dividend-free level of underlying index.

After collecting all options data available, the following data filters are employed: (1) observations with any missing data entry are discarded; (2) observations are removed if maturity is shorter than one week or longer than one year; (3) observations are included only if the daily total trading volume for the corresponding maturity is nonzero. Namely, the entire daily observations for a single maturity are discarded if none of them are traded; (4) observations are excluded if the bid price is zero or higher than the ask price; (5) observations are removed if they violate the no-arbitrage condition; (6) observations are excluded if the midpoint of bid and ask prices is less than 0.375; and (7) observations are discarded if the bid-ask spread is larger than the mid-point price. Table 2.1 reports some summary statistics of the filtered option price observations.

2.4. Impact of truncation

In this section, we investigate how truncation affects the implied moment estimators of Bakshi et al. (2003) using model-based generated option prices. Section 2.4.1 explains

Table 2.1: Sample properties of S&P 500 index options dataset

This table presents summary statistics of the S&P 500 index options dataset used in this thesis. The dataset spans an eleven year time period from January 2000 to December 2010. All statistics are collected after applying the following filtration conditions: (1) Observations with any missing data entry are removed; (2) Observations are excluded if time to maturity is shorter than one week or longer than one year; (3) Observations are included only if sum of daily trading volume for the corresponding maturity date is nonzero. In other words, the entire daily observations for a maturity date are excluded if none of them are traded in that day; (4) Observations are removed if the bid price is zero or higher than the ask price; (5) Observations that violate the no-arbitrage restriction of upper and lower bounds are excluded; (6) Observations with bid-ask mid-point price less than 0.375 are removed; and (7) Observations with bid-ask spread larger than mid-point price are excluded.

Moneyness category	Moneyness ($m = K/S$)		Calls: # of trading days to expiration (t) $5 \le t < 42$	$42 \le t < 126$	$126 \le t \le 252$	Sub-total # of obs.	Puts: # of trading days to expiration (t) $5 \le t < 42$	$42 \le t < 126$	$126 \le t \le 252$	Sub-total # of obs.
ITM calls, OTM puts	$m < 0.70$	Price	462.21	473.66	469.76		1.29	2.14	5.17	
		Eff. spread	1.37	1.35	1.42		0.42	0.42	0.57	
		BS implied vol.	0.9287	0.4874	0.3317		0.6358	0.4644	0.3847	
		# of obs.	7,658	7,027	8,153	22,838	7,592	24,873	27,541	60,006
	$0.70 \le m < 0.85$	Price	242.76	248.22	279.95		2.42	6.58	17.70	
		Eff. spread	1.35	1.36	1.39		0.40	0.57	0.92	
		BS implied vol.	0.4800	0.3199	0.2639		0.4097	0.3247	0.2707	
		# of obs.	26,402	34,399	30,033	90,834	35,449	47,189	31,580	114,218
	$0.85 \le m < 1.00$	Price	88.05	110.26	145.87		9.78	25.63	47.75	
		Eff. spread	1.17	1.27	1.34		0.58	0.98	1.23	
		BS implied vol.	0.2496	0.2361	0.2227		0.2465	0.2390	0.2266	
		# of obs.	103,886	75,313	44,697	223,896	108,908	75,536	44,691	229,135
OTM calls, ITM puts	$1.00 \le m < 1.15$	Price	9.77	21.95	49.28		68.32	91.07	117.48	
		Eff. spread	0.59	0.90	1.21		1.24	1.33	1.37	
		BS implied vol.	0.1875	0.1861	0.1866		0.2117	0.2002	0.2012	
		# of obs.	80,673	73,663	42,739	197,075	73,099	62,471	35,690	171,260
	$1.15 \le m < 1.30$	Price	2.49	3.98	12.69		214.17	216.81	239.29	
		Eff. spread	0.57	0.55	0.82		1.62	1.70	1.72	
		BS implied vol.	0.3303	0.2074	0.1800		0.3866	0.2377	0.2106	
		# of obs.	6,791	21,343	25,948	54,082	15,275	15,379	13,443	44,097
	$m \ge 1.30$	Price	1.21	1.87	3.59		442.01	474.16	484.41	
		Eff. spread	0.59	0.55	0.57		1.71	1.79	1.94	
		BS implied vol.	0.4665	0.2843	0.2187		0.6716	0.3632	0.2705	
		# of obs.	892	5,182	15,984	22,058	10,669	13,640	12,704	37,013
Total # of obs.						610,783				655,729
							1,266,512			

how the option prices are generated. Section 2.4.2 investigates the impact of truncation on implied moment estimators.

2.4.1 Generation of option prices

In this thesis, two sets of OTM option prices are generated using the Black-Scholes constant volatility (BS) model and stochastic volatility and jump (SVJ) model of Bakshi et al. (1997), respectively. BS model is chosen in order to examine the case where the implied RND is symmetric, as well as to link this thesis to Dennis and Mayhew (2002) and Jiang and Tian (2005) in which BS model is employed. On the other hand, SVJ model is adopted in our analysis to generate a more realistic simulation setting, as in Jiang and Tian (2005).[1] For the SVJ case, based on Bakshi et al. (1997), the underlying price is assumed to follow a process

$$\frac{dS(t)}{S(t)} = [r - \lambda\mu_J]dt + \sqrt{V(t)}d\omega_S(t) + J(t)dq(t), \tag{2.21}$$

$$dV(t) = [\theta_v - \kappa_v V(t)]dt + \sigma_v\sqrt{V(t)}d\omega_v(t), \tag{2.22}$$

$$\ln[1 + J\,(t)] \sim N(\ln[1 + \mu_J] - 0.5\sigma_J^2, \sigma_J^2), \tag{2.23}$$

where r is the constant risk-free interest rate, λ is the frequency of jumps per year, $V(t)$ is the part of return variance that is due to diffusion process, $\omega_S(t)$ and $\omega_v(t)$ are standard Brownian motions with $\mathrm{Cov}[d\omega_S(t), d\omega_v(t)] = \rho dt$, $J(t)$ is the percentage jump size that is lognormally i.i.d. over time as in (2.23), μ_J is the unconditional mean of $J(t)$, $\sigma_J(t)$ is the standard deviation of $\ln[1 + J(t)]$, $q(t)$ is a Poisson jump counter with intensity λ so that $P[dq(t) = 1] = \lambda dt$ and $P[dq(t) = 0] = (1 - \lambda)dt$, κ_v is the speed of adjustment of $V(t)$, θ_v/κ_v is the long-term mean of $V(t)$, and σ_v is the volatility of $V(t)$.

In order to calculate the option prices based on the two models, we first need to assign the values of the parameters in the pricing formulas of BS and SVJ models. While the price volatility σ is the only parameter in BS model, there are eight parameters

[1] Although model-dependence is a thing that should be avoided in this thesis, option pricing models are still used in this chapter to give the readers an intuition about how the option prices, the Black-Scholes implied volatility curve, and the implied risk-neutral density are shaped in the markets, and how the shapes are different from those in the Black-Scholes world. Models should be used for this purpose because the true value of implied moments are required to demonstrate the size of truncation error and, at the same time, the true values should be realistic as well as reasonable. Given these reasons, the option pricing models in this chapter should be regarded as auxiliary tools that are used to generate the necessary ingredients for an illustration of truncation error, not as main topics on which a further analysis needs to be conducted.

$\{\kappa_v, \theta_v, V, \sigma_v, \mu_J, \sigma_J, \rho, \lambda\}$ in SVJ model. To be more realistic and provide a foundation for our further empirical investigation, we calibrate the model parameters using daily S&P 500 index options data. Following Bakshi et al. (1997), if there are n call and m put prices that are observed on day t, for each of BS and SVJ models we find the parameter vector $\boldsymbol{\phi}(t)$ which solves

$$\min_{\boldsymbol{\phi}(t)} \left[\sum_{i=1}^{n} |C^*(t, \tau_i; K_i) - C(t, \tau_i; K_i)|^2 + \sum_{j=1}^{m} |P^*(t, \tau_j; K_j) - P(t, \tau_j; K_j)|^2 \right], \tag{2.24}$$

where $C^*(t, \tau_i; K_i)$ and $C(t, \tau_i; K_i)$ are the observed and model prices of ith call option with time to maturity τ_i and strike price K_i, $P^*(t, \tau_j; K_j)$ and $P(t, \tau_j; K_j)$ are the observed and model prices of jth put option with time to maturity τ_j and strike price K_j, respectively.[2] We then calculate the mean of daily parameter vectors for the entire T trading days in our sample period, which can be defined as

$$\boldsymbol{\Phi} \equiv \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{\phi}(t). \tag{2.25}$$

This mean vector $\boldsymbol{\Phi}$ for each model is employed as the parameter vector to calculated model option prices. The model OTM option prices are calculated while fixing the time to maturity to be three months. The elements in $\boldsymbol{\Phi}$ are listed in the first column of Table 2.2, with some other related variables and summary statistics. It is shown in Table 2.2 that SVJ model can explain the option price sample much better than BS model does, which is consistent with Bakshi et al. (1997) and several other related studies.

Figure 2.1 illustrates some basic properties of model OTM option prices. In Figures 2.1a and 2.1b, it can be found that OTM put and some near-the-money (NTM) calls are more expensive in SVJ case, whereas most of OTM calls are more expensive in BS case. Accordingly, when converted to a Black-Scholes implied volatility curve, SVJ option prices show a volatility skew whose left tail is above the flat implied volatility curve of BS options prices but the right tail is below the flat curve, as in Figure 2.1c. Zhang and Xiang (2008) show that there is a positive relationship between the slope of implied volatility curve and the skewness of the implied RND, as well as between the curvature of implied

[2]See the appendix of Bakshi et al. (1997) to find the characteristic functions that constitute a closed-form solution for model call price in SVJ case. In this thesis, model price for puts are calculated using the put-call parity relationship.

volatility curve and the kurtosis of the implied RND. These relationships suggest that the implied RND is negatively skewed and leptokurtic in the SVJ case.

Table 2.2: Summary statistics of daily calibration result

This table presents summary statistics of the daily calibration results that are used for setting the model parameter values and generating option prices. BS parameter calibration result is presented in Panel A. SVJ parameter calibration result can be found in Panel B. Panel C compares the squared error of calibration results. Panel D provides information about the other variables that are used for generating option prices.

	Mean	Standard error	5^{th} percentile	25^{th} percentile	Median	75^{th} percentile	95^{th} percentile
Panel A. Black-Scholes (BS) model parameter							
σ	0.2033	0.0622	0.1249	0.1530	0.2020	0.2315	0.3316
Panel B. Stochastic volatility and jump (SVJ) model parameters							
κ_v	4.4592	1.8378	2.0605	2.9532	4.2420	5.4808	8.0536
θ_v	0.2108	0.1248	0.0641	0.1220	0.1833	0.2662	0.4585
V_0	0.0494	0.0600	0.0088	0.0176	0.0342	0.0572	0.1422
σ_v	0.8116	0.3603	0.3564	0.5504	0.7700	0.9764	1.4665
μ_J	−0.1000	0.1301	−0.3193	−0.1964	−0.0886	−0.0006	0.0933
σ_J	0.1546	0.1136	0.0144	0.0555	0.1341	0.2358	0.3609
ρ	−0.6812	0.1114	−0.8718	−0.7566	−0.6760	−0.6044	−0.5060
λ	0.1583	0.1398	0.0150	0.0593	0.1207	0.2132	0.4562
Panel C. Squared error (SE)							
Sum of SE (BS)	16514.51	20298.67	3262.96	5209.51	8742.31	21507.79	49264.25
Sum of SE (SVJ)	876.24	3131.26	80.92	198.60	425.21	845.74	2576.11
SE per option (BS)	30.85	22.67	11.49	17.02	24.92	39.16	62.34
SE per option (SVJ)	1.98	8.29	0.26	0.54	0.96	1.90	6.11
Panel D. Other variables							
# of options	460.05	242.88	236	283	332	633	948
S&P 500 index	1183.21	189.91	860.02	1068.13	1179.21	1324.97	1491.56
3-month risk-free rate	0.0301	0.0205	0.0031	0.0119	0.0263	0.0505	0.0671
3-month dividend rate	0.0167	0.0051	0.0073	0.0137	0.0182	0.0205	0.0231

Figure 2.2 illustrates the weight functions and weighted option prices. Here, the term *weight functions* refers to the rational functions that are used to multiply the OTM option prices to get the fair values of V, W, and X, i.e., the second derivative H_{SS} of the payoff functions that are defined in Equations (2.9)–(2.11). Figure 2.2a depicts how the weight functions assign weights to OTM option prices along the strike price domain. In Figure 2.2a, two notable points can be found. First, all the three weight functions assign much heavier weights to OTM puts than OTM calls. Second, the line for volatility contract V is relatively flat, whereas those for cubic contract W and quartic contract X becomes

steeper as the strike price decreases. This implies that DOTM put options have relatively larger effect for implied skewness and kurtosis estimators.

Figure 2.2b-2.2d illustrate the weighted OTM option prices which construct the fair value of V,W, and X, respectively. Although Figure 2.2b maintains the shape that is similar to Figure 2.1a, it can be found in Figures 2.2c and 2.2d that the weighted option prices are in a totally different shape when compared to the unweighted prices. In Figure 2.2c and 2.2d, weighted price converges to zero as the strike price approaches to underlying price, and smaller absolute weight is assigned to NTM options than in Figure 2.2b. Accordingly, moderately OTM options have the highest absolute weighted price, and it is also relatively higher for DOTM options as well. This implies that OTM options with little impact on the fair value of V can have a significant impact on the fair value of W and X. This is even more evident in SVJ case, where OTM puts are significantly expensive. In addition, the weighted price of OTM calls and puts are shown to have the opposite sign to each other in Figure 2.2c. This is due to the fact that log-moneyness $\ln(S(t)/K)$ is negative when $K < S(t)$. This shows that it is the difference between, not the sum of, weighted call and put prices that determines the fair value of W.

Given that the fair values of W and X are more closely related to the implied skewness and kurtosis estimators than the implied volatility estimator, it can be conjectured that DOTM options that have relatively minor effect on the implied volatility estimator can have a considerable impact on the implied skewness and kurtosis estimators. Since truncation is equivalent to the non-existence of DOTM option prices, this also implies that truncation will affect the implied skewness and kurtosis estimators more significantly than the implied volatility estimator.

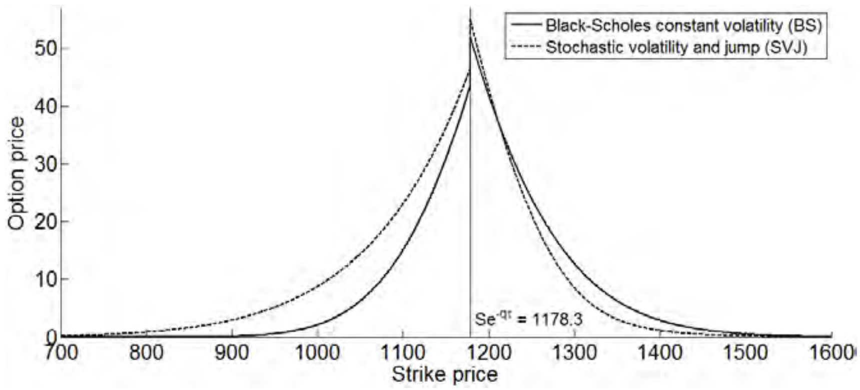
2.4.2 Impact of truncation on implied moment estimators

This subsection investigates how the implied moment estimators are affected by truncation. Since truncation occurs at both sides of the integration domain, there is a need to examine the impact of truncation on each side separately. To do this, we set two cases where the minimum and maximum strike prices vary in different ways. In the first case, the maximum strike price $K_{\max}$ is set to have one of the values $\{1190, 1290, 1390, 1490, 1590\}$ while the minimum strike price $K_{\min}$ is let to have any value between 670 and 1170 given the strike price interval of 0.1. In the second case, $K_{\min}$ is set to have one of the values

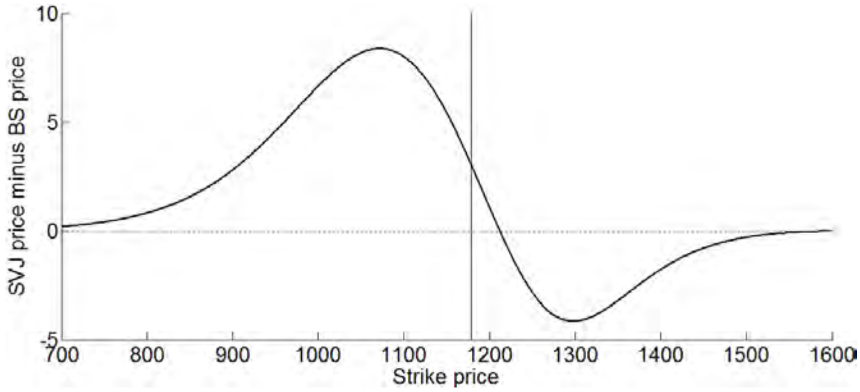
$\{770, 870, 970, 1070, 1170\}$ while $K_{\max}$ is let to have any value between 1190 and 1690 given the strike price interval. For each model, the true level of implied moments are

Figure 2.1: Option price properties

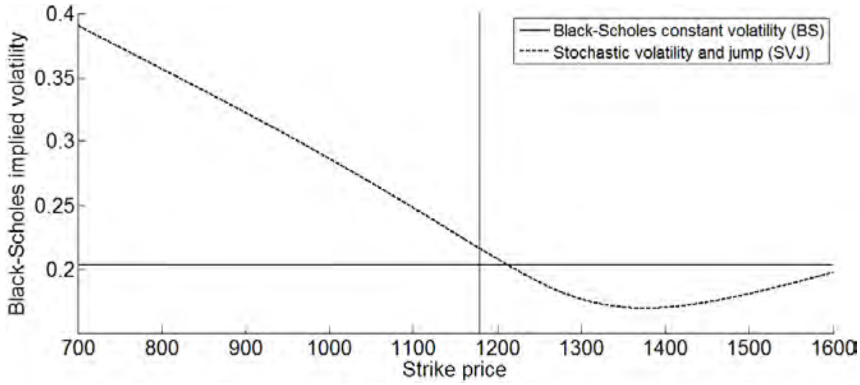
This figure illustrates some main properties of the model-based generated option prices that are used in this thesis. The OTM option prices are generated using two option pricing models, i.e., BS and SVJ models, and the model parameters that are set based on the model calibration results in Table 2.2. Underlying price, risk-free rate, and dividend rate are set as the sample mean. Time to maturity is set as three months. Figure 2.1a shows the OTM option price level for the strike price domain [700, 1600]. Figure 2.1b visualises the option price difference between the two models. Figure 2.1c demonstrates the Black-Scholes implied volatility curve for the two models. The curves are obtained by calculating the level of volatility with which the Black-Scholes option pricing formula derives the OTM option prices in Figure 2.1a.



(a) Option prices



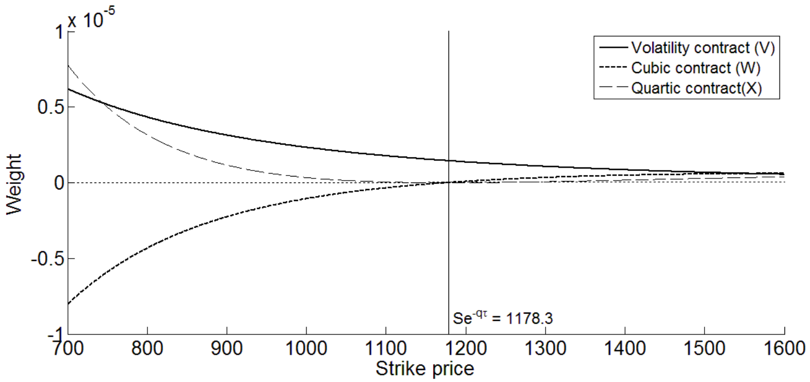
(b) Difference between option prices



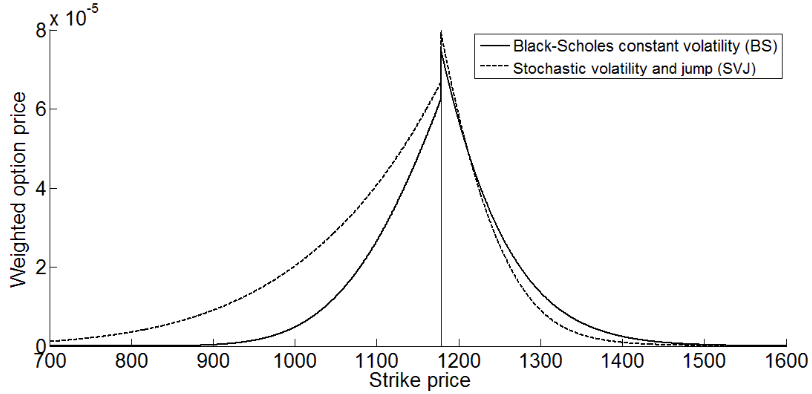
(c) Black-Scholes implied volatility curve

Figure 2.2: Weight functions and weighted option prices

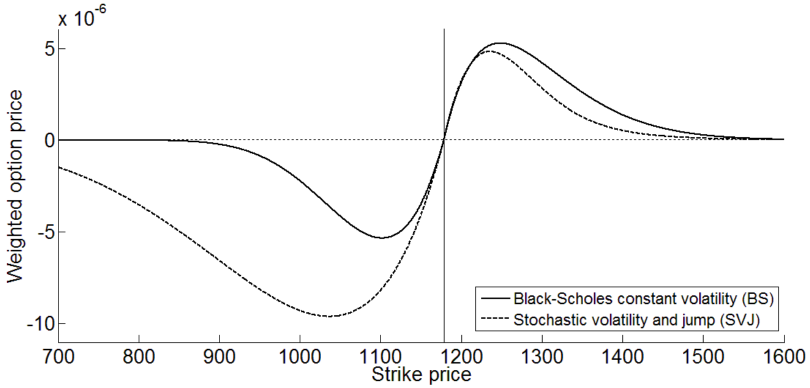
This figure illustrates the characteristics of weight functions that are used to determine the fair value of volatility, cubic, and quartic contracts of which the implied moment estimators of Bakshi et al. (2003) are nonlinear functions. Figure 2(a) shows the value of weight functions for strike price domain $[700, 1600]$. Figure 2(b)–2(d) demonstrate option price level after the prices are weighted by each of three weight functions, respectively. The options are OTM calls for the strike prices that are larger than the discounted underlying price, and OTM puts for the strike prices that are smaller than the discounted underlying price.



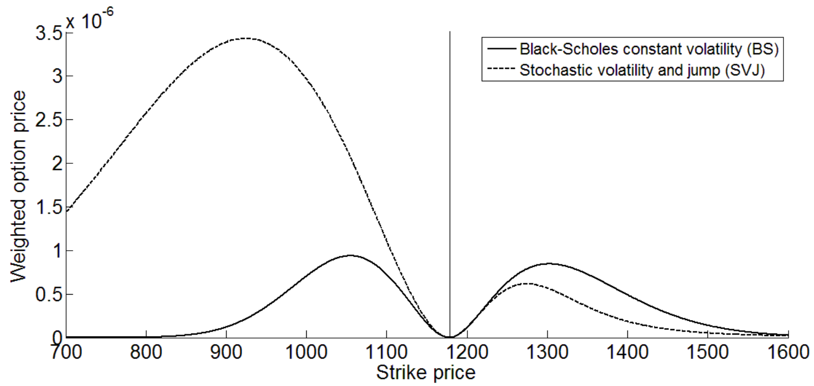
(a) Weight functions



(b) Weighted prices for volatility contract V



(c) Weighted prices for cubic contract W



(d) Weighted prices for quartic contract X

approximated using the corresponding option prices for the strike price domain $[S/3, 3S]$. The strike price interval is fixed as 0.1, which is small enough to make the strike price discreteness error negligible. For BS model, the true level of implied volatility, skewness, and kurtosis are 0.2032, 0, and 3. For SVJ model, the levels are approximated as 0.2387, -1.5156, and 6.8146, respectively.

Figure 2.3 illustrates the level of implied moment estimates for different values of $K_{\min}$ and $K_{\max}$ in BS case. In Figure 2.3, some notable points can be found. First, impact of marginal change in $K_{\min}$ or $K_{\max}$ decreases as they become further from the underlying price. This is intuitive given that option price decreases rapidly as the option goes outer-of-the-money. Second, DOTM options are shown to have larger impact on implied kurtosis estimate, when compared to implied volatility or skewness estimate. The lines in Figures 2.3e and 2.3f are found to be curved all along the x-axis, whereas those in Figures 2.3a–2.3d become almost flat when the newly included or excluded option is extremely DOTM. Third, including OTM calls (puts) results in an increase (decrease) in implied skewness estimate, whereas including any OTM options will lead to an increase in implied volatility estimate. This is because the implied skewness estimate is significantly affected by the value of cubic contract W for which the weight value is positive (negative) for OTM calls (puts), whereas the implied volatility estimate is mostly determined by the value of volatility contract V for which the weight value is positive for every OTM option. Furthermore, the relationship between inclusion of additional OTM options and change in implied kurtosis estimate is shown to be more complicated. Overall, the results imply that truncation needs to be treated in a more sophisticated way when higher implied moments are estimated.

Figure 2.4 demonstrates the level of implied moment estimates for different values of $K_{\min}$ and $K_{\max}$ in SVJ case. When compared to Figure 2.3, two significant differences can be found in Figure 2.4. First, OTM puts have much larger impact on all three implied moment estimates than OTM calls. There are visible gaps among all the lines in Figures 2.4a, 2.4c, and 2.4e, and every line in Figures 2.4b, 2.4d, and 2.4f are shown to be curved all along the x-axis. Second, given the larger impact of OTM puts, it is more clearly shown that the impact of OTM puts becomes larger for higher implied moments. For instance, including or excluding OTM puts in strike price domain $[700, 900]$ leads to changes in implied volatility, skewness, and kurtosis estimates by $\pm 0.007, \pm 0.363$, and

± 2.542, whose magnitude are about 2.93, 23.95, and 37.30 percent of approximated true level, respectively.[3]

Given that the calibration result of SVJ model is more closely related to S&P 500 index options data than BS model is, it is likely that the result in Figure 2.4 have more realistic implications about impact of truncation on implied moment estimators. Hence, it can be conjectured that OTM puts have larger impact on implied moment estimators than OTM calls do, and the impact is even larger on higher moment estimators.

2.5. Conclusion

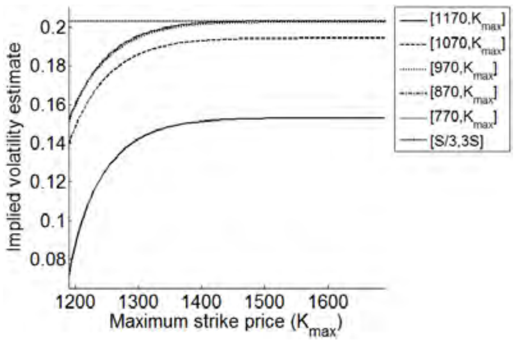
The model-free implied moment estimators of Bakshi et al. (2003) are obviously a set of ready-made econometric instruments that are easy and reliable to employ. This chapter provides an overview and conducts a theoretical analysis of the estimators in order to facilitate understanding the underpinnings of the estimation procedure. There are two notable findings in this chapter. First, the log-moneyness of the endpoints of the integration domain is closely related to the locations at which the implied RND is truncated. This finding will be utilised in later chapters to analyse the relationship between the truncation level and the truncation error size. Second, the size of truncation error is larger for the implied skewness and kurtosis estimators than the implied volatility estimator. This is the main issue that is addressed in this thesis and will be confirmed in later chapters.

In Chapters 3–5, we conduct a full-scale analysis of the impact of truncation on the implied moment estimators. In addition, we evaluate the effectiveness of the existing truncation error reduction methods, namely, LE and DSym, and propose DStab as an alternative method. During this process, the findings in this chapter will be mentioned frequently. Particularly, the implication of the log-moneyness of the endpoint strike prices is directly related to the new truncation error treatment method we propose here, and therefore, is worth enough to be considered as one of the main ideas in this thesis.

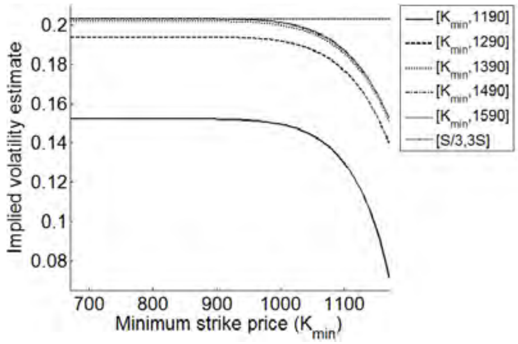
[3] The changes in the volatility, skewness, and kurtosis estimates that are made by excluding OTM options in price domain [700,900] are approximated using the mean of the changes for five different maximum strike prices.

Figure 2.3: Impact of truncation on implied moment estimates (BS)

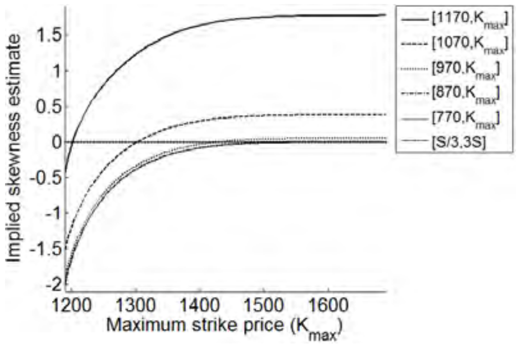
This figure illustrates the relationship between truncation and the level of implied moment estimates when the Black-Scholes constant volatility (BS) model is used for generating simulated option prices. In Figures 2.3a, 2.3c, and 2.3e, the maximum strike price $K_{\max}$ varies from 1,190 to 1,690, while the minimum strike price $K_{\min}$ is fixed at one of the values $\{770, 870, 970, 1070, 1170\}$. On the other hand, in Figures 2.3b, 2.3d, and 2.3f, $K_{\min}$ varies from 670 to 1,170, while $K_{\max}$ is fixed at one of the values $\{1190, 1290, 1390, 1490, 1590\}$. The straight line in each subfigure indicates the approximated true level of the corresponding implied moment, which is obtained by an estimation using the OTM option prices for the strike price domain $[3/S, 3S]$, where $S = 1178.3$ is the dividend-adjusted underlying price. The strike price interval is fixed at 0.1.



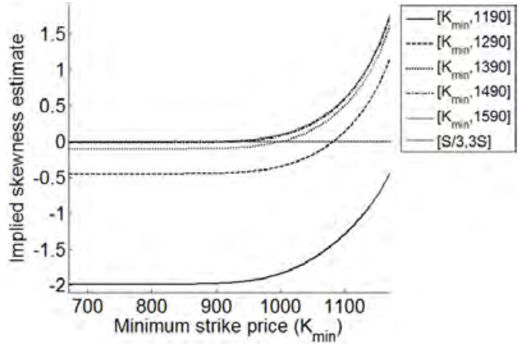
(a) Implied volatility estimate (fixed $K_{\min}$)



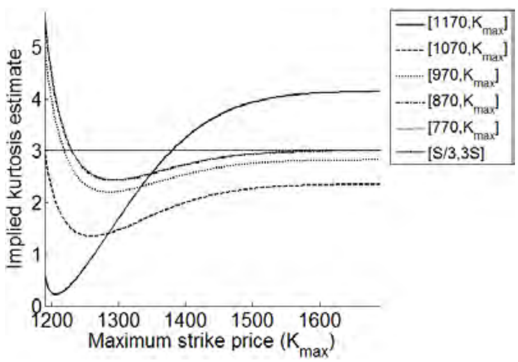
(b) Implied volatility estimate (fixed $K_{\max}$)



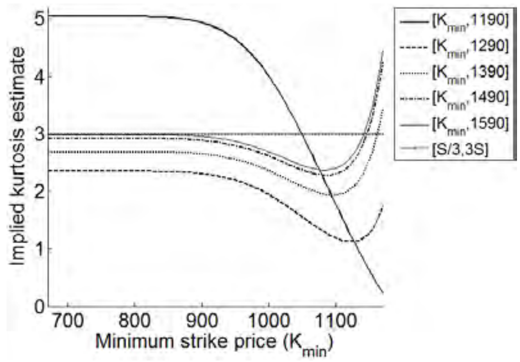
(c) Implied skewness estimate (fixed $K_{\min}$)



(d) Implied skewness estimate (fixed $K_{\max}$)



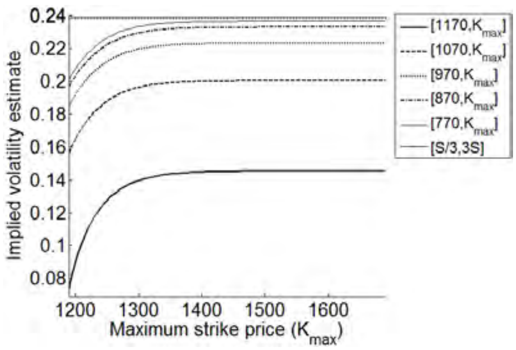
(e) Implied kurtosis estimate (fixed $K_{\min}$)



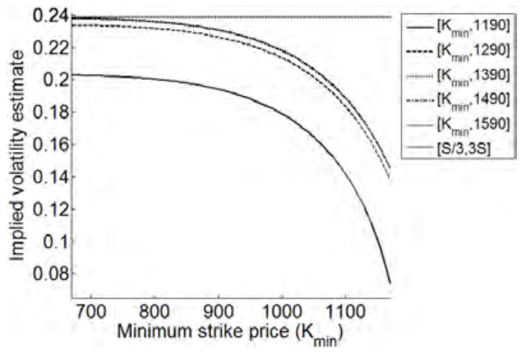
(f) Implied kurtosis estimate (fixed $K_{\max}$)

Figure 2.4: Impact of truncation on implied moment estimates (SVJ)

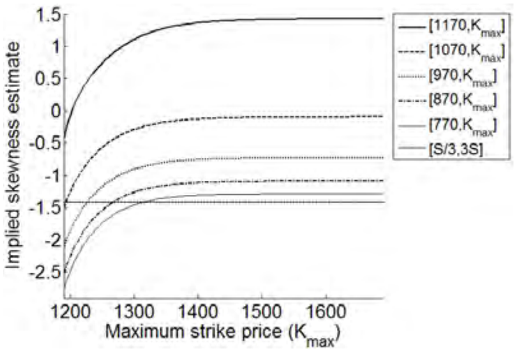
This figure illustrates the relationship between truncation and the level of implied moment estimates when the stochastic volatility and jump (SVJ) model is used for generating simulated option prices. In Figures 2.4a, 2.4c, and 2.4e, the maximum strike price $K_{\max}$ varies from 1,190 to 1,690, while the minimum strike price $K_{\min}$ is fixed at one of the values $\{770, 870, 970, 1070, 1170\}$. On the other hand, in Figures 2.4b, 2.4d, and 2.4f, $K_{\min}$ varies from 670 to 1,170, while $K_{\max}$ is fixed at one of the values $\{1190, 1290, 1390, 1490, 1590\}$. The straight line in each subfigure indicates the approximated true level of the corresponding implied moment, which is obtained by an estimation using the OTM option prices for the strike price domain $[3/S, 3S]$, where $S = 1178.3$ is the dividend-adjusted underlying price. The strike price interval is fixed at 0.1.



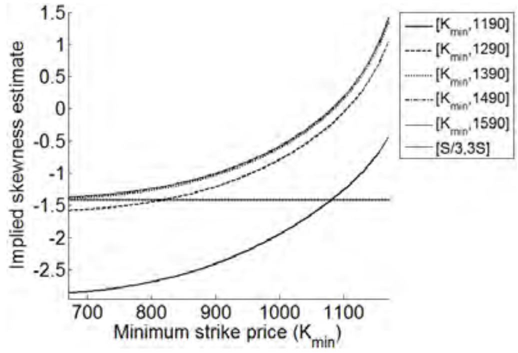
(a) Implied volatility estimate (fixed $K_{\min}$)



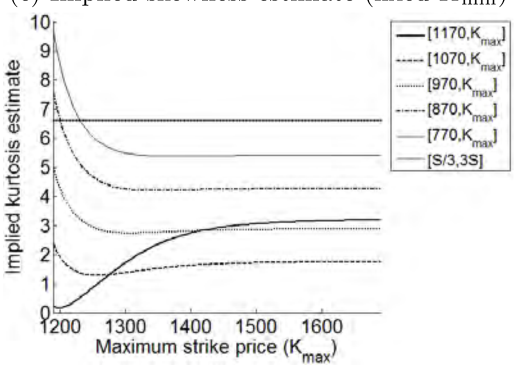
(b) Implied volatility estimate (fixed $K_{\max}$)



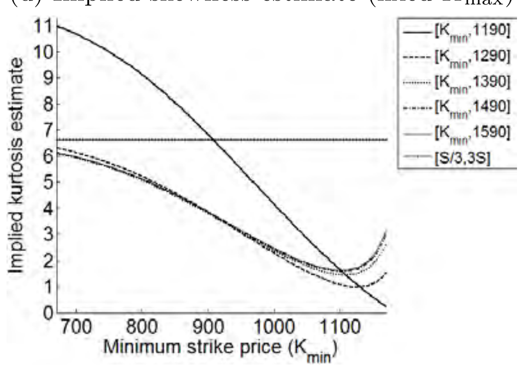
(c) Implied skewness estimate (fixed $K_{\min}$)



(d) Implied skewness estimate (fixed $K_{\max}$)



(e) Implied kurtosis estimate (fixed $K_{\min}$)



(f) Implied kurtosis estimate (fixed $K_{\max}$)

Chapter 3

Effectiveness of linear extrapolation in model-free implied moment estimation

Chapter Summary

This chapter shows that the sensitivity of the implied moment estimators of Bakshi et al. (2003) to a marginal change in option price availability can be formulated, regardless of whether LE is applied. Using S&P 500 index options data and sensitivity functions for the cases with and without LE applied, this chapter then investigates how effectively LE makes implied moment estimators less sensitive to option price unavailability. The empirical results suggest that LE is effective for all three estimators, although implied skewness and kurtosis estimators remain sensitive even with LE applied.

3.1. Introduction

The question of how unobservable DOTM option prices can be inferred has been raised and answered in several studies. Addressing this question is critical for the nonparametric estimation of the implied RND because these option prices contain crucial information on tail density. Although various option pricing models and sophisticated Black-Scholes implied volatility curve fitting schemes can answer this question, using one of these will cost the estimation procedure its model-freeness. With this limitation, LE has become one of the most popular DOTM option price approximation procedures for model-free implied moment estimation, because LE has little effect on model-freeness as a result of its approximation-based simple approach. Recently, with the increased popularity of the implied moment estimators of Bakshi et al. (2003), the use of LE has been extended and is mostly in conjunction with the implied skewness and kurtosis estimators of Bakshi et al. (2003) (e.g., Buss and Vilkov, 2012; Chang et al., 2012; Chang et al., 2013; DeMiguel et al., 2013; Neumann and Skiadopoulos, 2013).

LE is first introduced by Jiang and Tian (2005) to reduce the estimation error of model-free implied volatility estimator of Britten-Jones and Neuberger (2000). This method is conducted initially through the assumption that the Black-Scholes implied volatility curve is flat beyond the minimum and maximum strike prices of the orignial curve, followed by the extension of the curve on the basis of such an assumption. The missing DOTM option prices can then be inferred from this extended curve. In other words, LE can be regarded as a zeroth-order approximation of Black-Scholes implied volatility. As the order suggests, LE is a rough way of approximation. However, Jiang and Tian (2005) show that LE makes the estimation error smaller than that in the case in which no treatment is applied. From this case, LE can be conjectured to have at least a positive effect on reducing the estimation error of implied volatility estimator that is due to the unavailability of DOTM option prices. Jiang and Tian (2005) call this type of estimation error as a "truncation error" because such an error is caused by DOTM option prices that look as if they are "truncated."

However, two questions should be answered before applying LE, especially for implied skewness or kurtosis estimation. First, how effective is LE? In other words, is it adequately effective to make the truncation error negligible? The severity of truncation is different across markets and over time, so the truncation error can generate noise in the estimate if it is incompletely reduced. Hence, although determining whether LE can alleviate truncation error is by itself valuable, checking if LE can alleviate the truncation error up to the level in which the error becomes insignificant is also needed. Second, can LE be applied to higher moment estimators, i.e., the implied skewness and kurtosis estimators of Bakshi et al. (2003)? Although Jiang and Tian (2005) show the effectiveness of LE and derives the upper bound of truncation error for the implied volatility estimator of Britten-Jones and Neuberger (2000), neither the effectiveness nor the upper bound can be accepted as it is when LE is employed for another estimator. Furthermore, given that skewness and kurtosis are more closely related to the shape of the tail density than volatility is, the truncation error is likely larger for these higher moment estimators. Therefore, the effectiveness of LE should be assessed separately when it is used for the higher moment estimators of Bakshi et al. (2003).

This chapter proposes a measure with which the effectiveness of LE can be assessed for a set of European OTM options whose strike prices do not completely span the positive real line. A significant advantage of this measure is that it can be applied to option prices that are observed from markets. In most related studies, option prices are generated using option pricing models to demonstrate the truncation error (e.g., Dennis and Mayhew, 2002; Jiang and Tian, 2005; Dennis and Mayhew, 2009), given that the true level of the implied moment is required to calculate truncation error. With the limited availability of market option prices, obtaining the true level of implied moment from market option prices is impossible, and therefore, measuring the truncation error from them is also impossible. Hence, generated option prices have been used as an alternative. On the other hand, the problem is circumvented in this chapter through the use of a different approach to measure the effectiveness of LE.

If the truncation error is reduced effectively so that the effect of truncation on the implied moment estimate is negligible, the estimate should not change significantly after a marginal change in option price availability. In other words, if the implied moment estimate varies significantly when the moment is re-estimated after an inclusion or exclusion of a few option prices while LE is consistently applied, the truncation error has not been fully reduced by LE. Hence, the sensitivity of the implied moment estimate to a marginal change in option price availability can be used to assess the effectiveness of LE. This chapter shows that this sensitivity can be formulated for the estimators of Bakshi et al. (2003), regardless of whether LE is applied or not. Measuring the effectiveness of LE is therefore possible through a comparison of the sensitivity of implied moment estimates with and without the application of LE.

The following results are obtained through an empirical analysis of S&P 500 index options data. First, all three implied moment estimators of Bakshi et al. (2003) tend to become less sensitive to truncation after LE. This result implies that LE is indeed effective for all three estimators. With this result, LE can be conjectured to also reduce the truncation error of model-free implied moment estimators other than the implied volatility estimator of Britten-Jones and Neuberger (2000). Second, the implied volatility estimator of Bakshi et al. (2003) is found to be considerably robust to truncation when applied to S&P 500 index options data, regardless of whether LE is applied or not. Finally, the implied skewness and kurtosis estimators are found to be relatively sensitive to truncation, especially when the influence of LE on implied moment estimate is too strong.

Although the overall result suggests that LE is effective for all implied moment estimators of Bakshi et al. (2003), truncation error can also be conjectured to be too large to be regarded as negligible even after LE when implied skewness or kurtosis is estimated. This result implies that a supplementary truncation treatment may need to be considered along with LE when a higher implied moment is estimated. A possible supplementary method that is easy to employ is an additional data filtration, which removes observations with too high truncation sensitivity with LE. With this, the truncation sensitivity function is by itself found to be an instrument for truncation treatment.

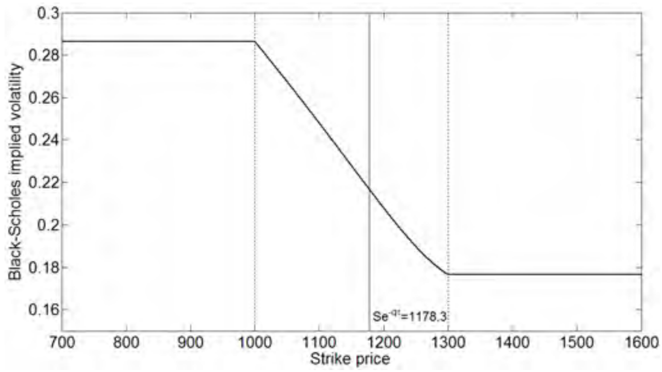
The rest of this chapter is organised as follows. Section 3.2 provides a brief description of LE. Section 3.3 explains how truncation sensitivity of implied moment estimators can be formulated for both cases with and without LE applied. Section 3.4 demonstrates how the sensitivity function defined in Section 3.3 can be applied to option prices. Section 3.5 reports the results of empirical analysis on S&P 500 index options data. Section 3.6 concludes this chapter.

3.2. Linear extrapolation

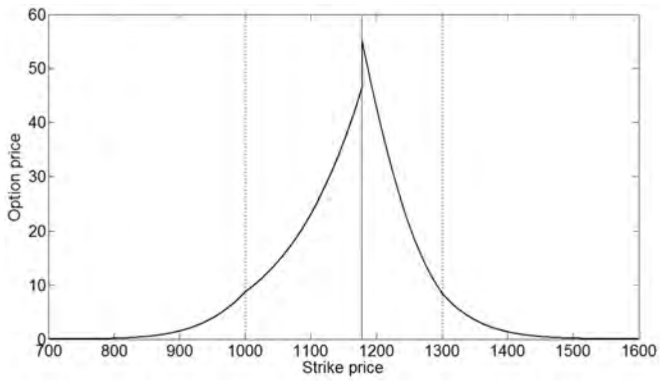
LE can be regarded as an approximation of order zero of Black-Scholes implied volatility with respect to strike price. To conduct LE, OTM option prices first need to be converted to Black-Scholes implied volatility values. Next, the level of Black-Scholes implied volatility at the minimum and maximum strike prices, for which the corresponding OTM option prices are observable, are collected. These implied volatility levels are then used to extend the implied volatility curve by assuming that the level of implied volatility is constant beyond the minimum and maximum strike prices. Finally, the extended implied volatility curve is converted back to a more complete set of OTM option prices. Figure 3.1 shows the consequences of LE for the SVJ case in Chapter 2 when the OTM option prices are truncated at the strike prices of 700 and 1300.

Figure 3.1: Illustration of linear extrapolation (LE)

This figure illustrates the shape of the Black-Scholes implied volatility curve and option price curve after LE. OTM option prices are taken from the SVJ case in Chapter 2, and are assumed to be truncated at the strike prices of 700 and 1300. Figure 3.1a demonstrates the shape of Black-Scholes implied volatility curve after LE. Figure 3.1b shows the OTM option price level after LE.



(a) Black-Scholes implied volatility curve after LE



(b) Option prices after LE

3.3. Truncation sensitivity function

This section demonstrates how the sensitivity of the implied moment estimators to a marginal change in the truncation level can be formulated. Section 3.3.1 explains how the definition of the implied moment estimators can be rearranged in terms of log-moneyness. Section 3.3.2 shows how sensitivity can be defined. Section 3.3.3 explains how truncation sensitivity should be measured when LE is applied.

3.3.1 Rearrangement of the implied moment estimators

If the parameters t and τ are removed for brevity, and λ denotes the log-moneyness $\ln(K/S(t))$, Equations (2.12)–(2.14) can be rearranged as

$$V = \int_0^\infty \frac{2\,(1-\lambda)}{Se^\lambda} C(\lambda) d\lambda + \int_{-\infty}^0 \frac{2\,(1-\lambda)}{Se^\lambda} P(\lambda) d\lambda, \tag{3.1}$$

$$W = \int_0^\infty \frac{6\lambda - 3\lambda^2}{Se^\lambda} C(\lambda) d\lambda + \int_{-\infty}^0 \frac{6\lambda - 3\lambda^2}{Se^\lambda} P(\lambda) d\lambda, \tag{3.2}$$

$$X = \int_0^\infty \frac{12\lambda^2 - 4\lambda^3}{Se^\lambda} C(\lambda) d\lambda + \int_{-\infty}^0 \frac{12\lambda^2 - 4\lambda^3}{Se^\lambda} P(\lambda) d\lambda. \tag{3.3}$$

With Equations (3.1)–(3.3), option prices are considered for the integration domain in terms of log-moneyness, not strike price. Not only does this rearrangement simplify the definitions and allow the integration domain to be more closely related to the domain of the RND function, but this also enables the easy definition of the minimum and maximum values of the integration domain as simple univariate functions, as will be shown below.

3.3.2 Definition of sensitivity

Now, suppose that the minimum and maximum values of the log-moneyness domain are finite, i.e., a truncation does exist, and the endpoint values are derived from differentiable univariate functions $f(\alpha)$ and $g(\alpha)$, which satisfy the conditions $f(\alpha) \leq 0$ and $g(\alpha) \geq 0$ for all α, respectively. If this is the case, the fair value estimates can be defined as univariate functions $\widehat{V}(\alpha), \widehat{W}(\alpha)$, and $\widehat{X}(\alpha)$ as follows:

$$\widehat{V}(\alpha) = \int_0^{g(\alpha)} \frac{2(1-\lambda)}{Se^{\lambda}} C(\lambda) d\lambda + \int_{f(\alpha)}^{0} \frac{2(1-\lambda)}{Se^{\lambda}} P(\lambda) d\lambda, \tag{3.4}$$

$$\widehat{W}(\alpha) = \int_0^{g(\alpha)} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} C(\lambda) d\lambda + \int_{f(\alpha)}^{0} \frac{6\lambda - 3\lambda^2}{Se^{\lambda}} P(\lambda) d\lambda, \tag{3.5}$$

$$\widehat{X}(\alpha) = \int_0^{g(\alpha)} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} C(\lambda) d\lambda + \int_{f(\alpha)}^{0} \frac{12\lambda^2 - 4\lambda^3}{Se^{\lambda}} P(\lambda) d\lambda. \tag{3.6}$$

Then, the implied volatility, skewness, and kurtosis estimates can also be obtained as univariate functions:

$$\widehat{\mathrm{VOL}}(\alpha) = [e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{1/2}, \tag{3.7}$$

$$\widehat{\mathrm{SKEW}}(\alpha) = \frac{e^{r\tau}\widehat{W}(\alpha) - 3\widehat{\mu}(\alpha)e^{r\tau}\widehat{V}(\alpha) + 2\widehat{\mu}(\alpha)^3}{[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{3/2}}, \tag{3.8}$$

$$\widehat{\mathrm{KURT}}(\alpha) = \frac{e^{r\tau}\widehat{X}(\alpha) - 4\widehat{\mu}(\alpha)e^{r\tau}\widehat{W}(\alpha) + 6e^{r\tau}\widehat{\mu}(\alpha)^2\widehat{V}(\alpha) - 3\widehat{\mu}(\alpha)^4}{[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}(\alpha)^2]^2}, \tag{3.9}$$

where

$$\widehat{\mu}(\alpha) = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}\widehat{V}(\alpha) - \frac{e^{r\tau}}{6}\widehat{W}(\alpha) - \frac{e^{r\tau}}{24}\widehat{X}(\alpha). \tag{3.10}$$

Given Equations (3.4)–(3.9), the derivative of $\widehat{\mathrm{VOL}}(\alpha)$, $\widehat{\mathrm{SKEW}}(\alpha)$, and $\widehat{\mathrm{KURT}}(\alpha)$ with respect to α can be formulated as

$$\widehat{\mathrm{VOL}}'(\alpha) = \frac{1}{2}[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{-1/2}\left(e^{r\tau}\widehat{V}'(\alpha) - 2\widehat{\mu}(\alpha)\widehat{\mu}'(\alpha)\right), \tag{3.11}$$

$$\widehat{\mathrm{SKEW}}'(\alpha) = \frac{\widehat{\Theta}_S(\alpha)\widehat{\Gamma}'_S(\alpha) - \widehat{\Gamma}_S(\alpha)\widehat{\Theta}'_S(\alpha)}{\widehat{\Theta}^2_S(\alpha)}, \tag{3.12}$$

$$\widehat{\mathrm{KURT}}'(\alpha) = \frac{\widehat{\Theta}_K(\alpha)\widehat{\Gamma}'_K(\alpha) - \widehat{\Gamma}_K(\alpha)\widehat{\Theta}'_K(\alpha)}{\widehat{\Theta}^2_K(\alpha)}, \tag{3.13}$$

where $\widehat{\Gamma}_S$ and $\widehat{\Gamma}_K$ denote the numerator of $\widehat{\mathrm{SKEW}}(\alpha)$ and $\widehat{\mathrm{KURT}}(\alpha)$, respectively, and $\widehat{\Theta}_S$ and $\widehat{\Theta}_K$ denote the denominator of $\widehat{\mathrm{SKEW}}(\alpha)$ and $\widehat{\mathrm{KURT}}(\alpha)$, respectively. Equations (3.11)–(3.13) are direct consequences of the chain rule and the quotient rule. Given

Equations (3.4)–(3.9), the derivatives in Equations (3.11)–(3.13) can be obtained as

$$\widehat{\Gamma}'_S(\alpha) = e^{r\tau}\widehat{W}'(\alpha) - 3e^{r\tau}\left(\widehat{V}(\alpha)\widehat{\mu}'(\alpha) + \widehat{\mu}(\alpha)\widehat{V}'(\alpha)\right) + 6\widehat{\mu}^2(\alpha)\widehat{\mu}'(\alpha), \tag{3.14}$$

$$\widehat{\Theta}'_S(\alpha) = \frac{3}{2}[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]^{1/2}\left(e^{r\tau}\widehat{V}'(\alpha) - 2\widehat{\mu}(\alpha)\widehat{\mu}'(\alpha)\right), \tag{3.15}$$

$$\widehat{\Gamma}'_K(\alpha) = e^{r\tau}\widehat{X}'(\alpha) - 4e^{r\tau}\left(\widehat{W}(\alpha)\widehat{\mu}'(\alpha) + \widehat{\mu}(\alpha)\widehat{W}'(\alpha)\right) \tag{3.16}$$

$$+ 6e^{r\tau}\left(2\widehat{\mu}(\alpha)\widehat{V}(\alpha)\widehat{\mu}'(\alpha) + \widehat{\mu}^2(\alpha)\widehat{V}'(\alpha)\right) - 12\widehat{\mu}^3(\alpha)\widehat{\mu}'(\alpha), \tag{3.17}$$

$$\widehat{\Theta}'_K(\alpha) = 2[e^{r\tau}\widehat{V}(\alpha) - \widehat{\mu}^2(\alpha)]\left(e^{r\tau}\widehat{V}'(\alpha) - 2\widehat{\mu}(\alpha)\widehat{\mu}'(\alpha)\right), \tag{3.18}$$

$$\widehat{V}'(\alpha) = g'(\alpha)\frac{2(1-g(\alpha))}{Se^{g(\alpha)}}C(g(\alpha)) - f'(\alpha)\frac{2(1-f(\alpha))}{Se^{f(\alpha)}}P(f(\alpha)), \tag{3.19}$$

$$\widehat{W}'(\alpha) = g'(\alpha)\frac{6g(\alpha)-3g^2(\alpha)}{Se^{g(\alpha)}}C(g(\alpha)) - f'(\alpha)\frac{6f(\alpha)-3f^2(\alpha)}{Se^{f(\alpha)}}P(f(\alpha)), \tag{3.20}$$

$$\widehat{X}'(\alpha) = g'(\alpha)\frac{12g^2(\alpha)-4g^3(\alpha)}{Se^{g(\alpha)}}C(g(\alpha)) - f'(\alpha)\frac{12f^2(\alpha)-4f^3(\alpha)}{Se^{f(\alpha)}}P(f(\alpha)), \tag{3.21}$$

$$\widehat{\mu}'(\alpha) = -e^{r\tau}\left(\frac{1}{2}\cdot\widehat{V}'(\alpha) + \frac{1}{6}\cdot\widehat{W}'(\alpha) + \frac{1}{24}\cdot\widehat{X}'(\alpha)\right). \tag{3.22}$$

3.3.3 Truncation sensitivity with LE

This subsection derives the sensitivity of the implied moment estimate to a change in option price availability when LE is applied. Because extremely DOTM options have an infinitesimal value and, therefore, excluding them have little effect on moment estimates, extrapolation is generally done only up to fixed limit points that are far enough from the at-the-money point.[1] This approach is also used in this chapter, and extrapolation is assumed to be done up to the points in which log-moneyness is equal to finite and fixed limit values $\lambda_{\min}$ and $\lambda_{\max}$, respectively. Both $\lambda_{\min}$ and $\lambda_{\max}$ are presumed to be significantly different from zero so that both $f(\alpha)$ and $g(\alpha)$ do not exceed the corresponding limit values. Now, suppose that the fair value of moment-related contracts V, W, and X is estimated with the use of a set of option prices that cover the log-moneyness domain $[f(\alpha), g(\alpha)]$, as in Equations (3.4)–(3.6), and the level of Black-Scholes implied volatility in this domain is defined by a function $\sigma_{\text{BS}}(\lambda)$ which is differentiable with respect to log-moneyness λ. If the two endpoint Black-Scholes implied volatilities $\sigma_{\text{BS}}(f(\alpha))$ and $\sigma_{\text{BS}}(g(\alpha))$ are extrapolated

[1] Buss and Vilkov (2012) generate option prices up to the points in which the moneyness is 1/3 and 3, respectively. On the other hand, Neumann and Skiadopoulos (2013) choose the points where the option delta is 0.01 and 0.99, respectively.

up to the points in which log-moneyness is equal to $\lambda_{\min}$ and $\lambda_{\max}$, respectively, the fair value estimates for the volatility, cubic, and quartic contracts in Equations (3.4)–(3.6) are replaced with the new estimates $\widetilde{V}(\alpha)$, $\widetilde{W}(\alpha)$, and $\widetilde{X}(\alpha)$, which can be obtained as follows:

$$\begin{aligned}\widetilde{V}(\alpha) &= \widehat{V}(\alpha) + \int_{g(\alpha)}^{\lambda_{\max}} \frac{2(1-\lambda)}{Se^{\lambda}} \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda) d\lambda \\ &\quad + \int_{\lambda_{\min}}^{f(\alpha)} \frac{2(1-\lambda)}{Se^{\lambda}} \widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda) d\lambda, \qquad (3.23)\\ \widetilde{W}(\alpha) &= \widehat{W}(\alpha) + \int_{g(\alpha)}^{\lambda_{\max}} \frac{6\lambda-3\lambda^2}{Se^{\lambda}} \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda) d\lambda \\ &\quad + \int_{\lambda_{\min}}^{f(\alpha)} \frac{6\lambda-3\lambda^2}{Se^{\lambda}} \widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda) d\lambda, \qquad (3.24)\end{aligned}$$

$$\begin{aligned}\widetilde{X}(\alpha) &= \widehat{X}(\alpha) + \int_{g(\alpha)}^{\lambda_{\max}} \frac{12\lambda^2-4\lambda^3}{Se^{\lambda}} \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda) d\lambda \\ &\quad + \int_{\lambda_{\min}}^{f(\alpha)} \frac{12\lambda^2-4\lambda^3}{Se^{\lambda}} \widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda) d\lambda, \qquad (3.25)\end{aligned}$$

where

$$\begin{aligned}\widetilde{C}(\sigma, \lambda) &= S(N(d_1(\sigma, \lambda)) - e^{-\lambda r\tau} N(d_2(\sigma, \lambda))), \qquad (3.26)\\ \widetilde{P}(\sigma, \lambda) &= S(e^{-\lambda r\tau} N(-d_2(\sigma, \lambda)) - N(-d_1(\sigma, \lambda))), \qquad (3.27)\\ d_1(\sigma, \lambda) &= \frac{-\lambda + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad (3.28)\end{aligned}$$

$$d_2(\sigma, \lambda) = d_1(\sigma, \lambda) - \sigma\sqrt{\tau}, \qquad (3.29)$$

and $N(\cdot)$ denotes the standard normal cumulative distribution function. With this, the following proposition shows how the derivatives $\widetilde{V}'(\alpha)$, $\widetilde{W}'(\alpha)$, and $\widetilde{X}'(\alpha)$ can be formulated:

Proposition 3.1. If (1) the level of Black-Scholes implied volatility in log-moneyness domain $[f(\alpha), g(\alpha)]$ is defined by a function $\sigma_{\mathrm{BS}}(\lambda)$ which is differentiable with respect to log-moneyness λ, (2) the two Black-Scholes implied volatilities $\sigma_{\mathrm{BS}}(f(\alpha))$ and $\sigma_{\mathrm{BS}}(g(\alpha))$ are linearly extrapolated up to the point in which log-moneyness is equal to fixed limit values $\lambda_{\min}$ and $\lambda_{\max}$, respectively, and (3) the value of $\lambda_{\min}$ and $\lambda_{\max}$ is significantly different from zero so that both $f(\alpha)$ and $g(\alpha)$ are not supposed to exceed those limit values, then the derivatives $\widetilde{V}'(\alpha)$, $\widetilde{W}'(\alpha)$, and $\widetilde{X}'(\alpha)$ can be formulated as

$$\begin{aligned}\widetilde{V}'(\alpha) = \gamma_0 &\int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda)\exp\left[\gamma_1\lambda^2+\gamma_2\lambda+\gamma_3\right]\right) d\lambda \\ &+ \delta_0 \int_{\lambda_{\min}}^{f(\alpha)} \left(2(1-\lambda)\exp\left[\delta_1\lambda^2+\delta_2\lambda+\delta_3\right]\right) d\lambda, \end{aligned} \tag{3.30}$$

$$\begin{aligned}\widetilde{W}'(\alpha) = \gamma_0 &\int_{g(\alpha)}^{\lambda_{\max}} \left((6\lambda-3\lambda^2)\exp\left[\gamma_1\lambda^2+\gamma_2\lambda+\gamma_3\right]\right) d\lambda \\ &+ \delta_0 \int_{\lambda_{\min}}^{f(\alpha)} \left((6\lambda-3\lambda^2)\exp\left[\delta_1\lambda^2+\delta_2\lambda+\delta_3\right]\right) d\lambda, \end{aligned} \tag{3.31}$$

$$\begin{aligned}\widetilde{X}'(\alpha) = \gamma_0 &\int_{g(\alpha)}^{\lambda_{\max}} \left((12\lambda^2-4\lambda^3)\exp\left[\gamma_1\lambda^2+\gamma_2\lambda+\gamma_3\right]\right) d\lambda \\ &+ \delta_0 \int_{\lambda_{\min}}^{f(\alpha)} \left((12\lambda^2-4\lambda^3)\exp\left[\delta_1\lambda^2+\delta_2\lambda+\delta_3\right]\right) d\lambda, \end{aligned} \tag{3.32}$$

where

$$\gamma_0 = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha), \tag{3.33}$$

$$\gamma_1 = -\frac{1}{2\sigma_{\mathrm{BS}}(g(\alpha))^2\tau}, \tag{3.34}$$

$$\gamma_2 = \frac{r}{\sigma_{\mathrm{BS}}(g(\alpha))^2} - \frac{1}{2}, \tag{3.35}$$

$$\gamma_3 = -\frac{(r+0.5\sigma_{\mathrm{BS}}(g(\alpha))^2)^2\tau}{2\sigma_{\mathrm{BS}}(g(\alpha))^2}, \tag{3.36}$$

$$\delta_0 = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\mathrm{BS}}(f(\alpha)) \cdot f'(\alpha), \tag{3.37}$$

$$\delta_1 = -\frac{1}{2\sigma_{\mathrm{BS}}(f(\alpha))^2\tau}, \tag{3.38}$$

$$\delta_2 = \frac{r}{\sigma_{\mathrm{BS}}(f(\alpha))^2} - \frac{1}{2}, \tag{3.39}$$

$$\delta_3 = -\frac{(r+0.5\sigma_{\mathrm{BS}}(f(\alpha))^2)^2\tau}{2\sigma_{\mathrm{BS}}(f(\alpha))^2}, \tag{3.40}$$

$\sigma'_{\mathrm{BS}}(\lambda)$ is the derivative of Black-Scholes implied volatility function $\sigma_{\mathrm{BS}}(\lambda)$ with respect to log-moneyness λ, r is the risk-free rate, and τ is the time to maturity.

Proof. See Appendix A. □

3.4. Methodology

This section demonstrates how the truncation sensitivity funcitions introduced in Section 3.3 can be employed on option prices data. Section 3.4.1 explains how the Black-Scholes implied volatility curve is constructed in this chapter to obtain a dense set of option prices, as well as a differentiable Black-Scholes implied volatility function with respect to log-moneyness. In Section 3.4.2, the empirical procedure of truncation sensitivity estimation is described. Section 3.4.3 shows how the value of truncation sensitivity can be interpreted and used to assess the effectiveness of LE.

3.4.1 Construction of Black-Scholes implied volatility curve

An implied volatility curve is required for each maturity for which truncation sensitivity is measured, in order to obtain an adequate number of option prices from which a continuum of option prices can be approximated, as well as to estimate a differentiable implied volatility function. As done in Jiang and Tian (2005), maturity is first fixed to avoid the telescoping problem that is pointed out by Christensen et al. (2002). Black-Scholes implied volatilities are first collected from all available OTM option prices to fix the maturity. Next, a bicubic spline function is estimated with the use of the implied volatility observations. For the regions where some of the observations required for estimation are not available because of a difference in minimum or maximum strike price between different maturities, LE is applied to approximate the missing observation. With the estimated bicubic spline function, the implied volatility levels are then approximated at some fixed maturities for the strike prices for which at least one observation exists on that day. When only the exactly monthly maturities are considered, consecutive daily observations can be fully obtained only for the maturities of two, three, and four months during the sample period due to data filtration and liquidity issues. Given this limitation, implied volatility curves only for the maturities of two and four months are examined.

After the maturity is fixed, the implied volatility curves are constructed for each maturity. First, the implied volatility levels are additionally approximated for the minimum and maximum values of the strike price domain with the use of the bicubic spline function. If the minimum and maximum strike prices are not observable for a maturity, they are linearly approximated with the use of the corresponding endpoint strike prices for the two closest maturities for which the endpoints are observable. After this additional approximation, all implied volatility values that are located beyond the minimum or maximum strike price are discarded. Not only does this reflect the level of truncation observed in the market, but this also removes the effect of LE that is conducted while fixing the maturity.

Next, piecewise quadratic function is used to estimate the shape of the implied volatility curve, in accordance with the approach of Broadie et al. (2007). Similar to the approach of Broadie et al. (2007), the following function is fitted:

$$\sigma_{BS}(\lambda) = \mathbf{1}_{\lambda \le 0}[a_2\lambda^2 + a_1\lambda + a_0] + \mathbf{1}_{\lambda > 0}[b_2\lambda^2 + a_1\lambda + a_0] + \varepsilon, \tag{3.41}$$

where $\sigma_{BS}(\lambda)$ is the level of Black-Scholes volatility at log-moneyness λ, and $\mathbf{1}_C$ is an indicator function whose value is one when condition C holds and zero otherwise. The piecewise function is defined as a function of log-moneyness λ to obtain the derivative $\sigma'_{BS}(\lambda)$. Although the implied volatility curve can also be estimated with the use of the cubic spline function, which provides a perfect fit as in Jiang and Tian (2005), a curve estimated with the use of the cubic spline function can be winding severely, so that the derivative of the implied volatility function may not be approximated stably. Comparatively, the derivative of the piecewise qudratic function is more stable, although the function itself does not ensure a perfect fit. Because the derivative of the implied volatility function at endpoints is significantly important in measuring sensitivity when LE is applied, the piecewise quadratic function is chosen to estimate implied volatility curve. After the implied volatility curve is estimated, the curve is translated into OTM option prices for the strike prices between the minimum and maximum values of the strike price domain, with a strike price interval of 0.1. When LE is employed for implied moment or truncation sensitivity estimation, it is applied to this curve up to the points in which the strike prices are equal to $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-free index level on day t. This setting means that $\lambda_{\min}$ and $\lambda_{\max}$ in Equations (3.23)–(3.32) are set as $-\ln 3$ and $\ln 3$, respectively. The strike price interval is again set as 0.1 for LE.

3.4.2 Measuring truncation sensitivity

Regardless of whether LE is applied or not, the basic procedure of truncation sensitivity measurement is identical. Specifically, one first sets the endpoint functions $f(\alpha)$ and

$g(\alpha)$, and calculates the contract fair value estimates and their derivatives with respect to α. Then, all the other required derivatives can be calculated correspondingly on the basis of the procedure shown in Section 3.3.2. The only difference between the truncation sensitivity estimation with and without LE is the method of obtaining the contract fair value estimates and their derivatives. This subsection therefore describes the procedure of truncation sensivity measurement that can be applied to both cases with and without LE, with some additional comments for the case with LE which is slightly more complicated.

Endpoint functions $f(\alpha)$ and $g(\alpha)$ need to be defined first to measure sensitivity. Given Equations (3.4)–(3.6), $f(\alpha)$ and $g(\alpha)$ must be defined in a way that there exists a real number $\bar{\alpha}$ for which $f(\bar{\alpha}) = \ln(K_{\min}/S)$ and $g(\bar{\alpha}) = \ln(K_{\max}/S)$, where $K_{\min}$ and $K_{\max}$ are the minimum and maximum strike prices of the integration domain, respectively, and S is the underlying price. A simple definition that is applicable to any observed data is $f(\alpha) = c\alpha$ and $g(\alpha) = \alpha$, where

$$c = \frac{\ln(K_{\min}/S)}{\ln(K_{\max}/S)}. \tag{3.42}$$

With this definition, two implicit assumptions are made. First, α is nonnegative, no option prices are available when $\alpha = 0$, and the option price availability increases as α becomes larger. Second, the log-moneyness ratio c is not changed by an increase or decrease in option price availability. The first assumption is acceptable because it makes the relationship between α and option price availability clearly defined. The second assumption is also reasonable because it ensures that option price availability increases at each side (call side or put side) as α becomes larger, while also satisfying the condition that the real number $\bar{\alpha}$ mentioned above exists. Given such assumptions, this simple definition is used in this chapter and is employed for empirical analysis.

After $f(\alpha)$ and $g(\alpha)$ are set, the contract fair value estimates and their derivatives at $\alpha = \bar{\alpha}$ need to be calculated. With the definition of $f(\alpha)$ and $g(\alpha)$ above, $\bar{\alpha} = \ln(K_{\max}/S)$. $\widehat{V}(\bar{\alpha})$, $\widehat{W}(\bar{\alpha})$, $\widehat{X}(\bar{\alpha})$, $\widehat{V}'(\bar{\alpha})$, $\widehat{W}'(\bar{\alpha})$, and $\widehat{X}'(\bar{\alpha})$ need to be estimated as in Section 3.3.2 when LE is not employed, whereas $\widetilde{V}(\bar{\alpha})$, $\widetilde{W}(\bar{\alpha})$, $\widetilde{X}(\bar{\alpha})$, $\widetilde{V}'(\bar{\alpha})$, $\widetilde{W}'(\bar{\alpha})$, and $\widetilde{X}'(\bar{\alpha})$ are required when LE is considered. The fair value estimates and derivatives without LE are relatively easier to obtain because they can be collected via the model-free implied moment estimation procedure of Bakshi et al. (2003). On the other hand, two extra subtleties exist when derivatives with LE are calculated. First, the derivative $\sigma'_{BS}(\lambda)$ of the Black-Scholes implied volatility function needs to be evaluated at $\lambda = f(\bar{\alpha})$ and $\lambda = g(\bar{\alpha})$. This evaluation can be done with the implied volatility curve that is approximated using a differentiable function, i.e., piecewise quadratic function, which is demonstrated in Section 3.4.1. Second, because Equations (3.30)–(3.32) are in the form of transcendental function, no analytic solution exists for them. Fortunately, they can still be evaluated numerically with the use of a computational software. *Wolfram Mathematica* is used in this chapter to evaluate $\widetilde{V}'(\bar{\alpha})$, $\widetilde{W}'(\bar{\alpha})$, and $\widetilde{X}'(\bar{\alpha})$. With all the fair value estimates and their derivatives with respect to α, now, the derivatives of the implied moment estimates with respect to α, i.e., the truncation sensitivity of the implied moment estimators, can be obtained. When LE is not considered, plugging in the value of the required variables into Equations (3.11)–(3.13) enables one to calculate the derivatives. It is also the case when LE is considered, but after replacing $\widehat{V}(\bar{\alpha})$, $\widehat{W}(\bar{\alpha})$, $\widehat{X}(\bar{\alpha})$, $\widehat{V}'(\bar{\alpha})$, $\widehat{W}'(\bar{\alpha})$, and $\widehat{X}'(\bar{\alpha})$ with $\widetilde{V}(\bar{\alpha})$, $\widetilde{W}(\bar{\alpha})$, $\widetilde{X}(\bar{\alpha})$, $\widetilde{V}'(\bar{\alpha})$, $\widetilde{W}'(\bar{\alpha})$, and $\widetilde{X}'(\bar{\alpha})$, respectively.

3.4.3 Interpretation of truncation sensitivity

Although truncation sensitivity provides information on how sensitive to truncation an implied moment estimator is, refining the information is still needed to assess the effectiveness of LE for two reasons. Firstly, unless no evidence exists that sensitivity is constant or at least stable over the α-axis, the measured sensitivity should be regarded as local and therefore be used to approximate a change in estimate caused by a small change in option price availability. Secondly, because option prices are assigned for discrete strike prices, defining the change in option price availability in terms of strike price rather than log-moneyness may be more practical. Hence, the change in implied moment estimate caused by an increase in strike price domain length by five, which is equal to the size of a single strike price interval in the S&P 500 index options market for short maturities, is linearly approximated to assess the effectiveness of LE.

To approximate this change, one first needs to determine the size of increase in α that is required to increase the strike price domain length by five. In other words, one needs to obtain the value of increment i, which satisfies

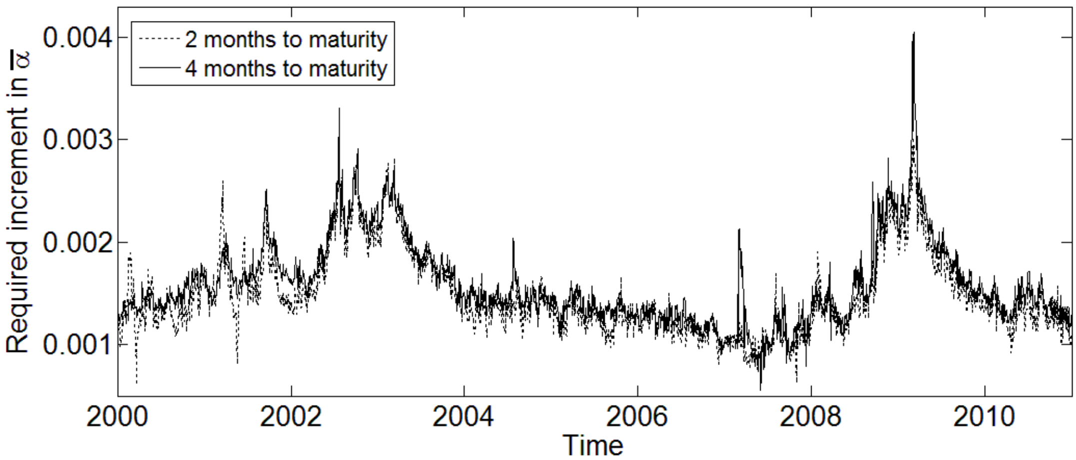
$$h(i) \equiv (Se^{g(\bar{\alpha}+i)} - Se^{f(\bar{\alpha}+i)}) - (Se^{g(\bar{\alpha})} - Se^{f(\bar{\alpha})}) - 5 = 0. \tag{3.43}$$

Figure 3.2: Increment in $\bar{\alpha}$ required to increase the strike domain length by five

This figure illustrates the level of increment in $\bar{\alpha}$, i.e., the value of α with which $f(\alpha)$ and $g(\alpha)$ coincide with the observed minimum and maximum strike prices, respectively, that is required to increase the strike domain length by five for the S&P 500 index options dataset used in this chapter. The dataset spans a time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months that are extracted from daily implied volatility surfaces. The value of the required increment i is approximated by minimising the absolute value of function $h(t, \tau, i)$ which is defined as

$$h(t, \tau, i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i)} - Se^{f(\bar{\alpha}(t,\tau)+i)}) - (S(t)e^{g(\bar{\alpha}(t,\tau))} - Se^{f(\bar{\alpha}(t,\tau))}) - 5,$$

where $S(t)$ is the dividend-adjusted underlying index level at day t, and $\bar{\alpha}(t, \tau)$ is the value of α with which $f(\alpha)$ and $g(\alpha)$ coincide with the observed minimum and maximum strike prices, respectively, for day t and maturity τ.



Given the definition of $f(\alpha)$ and $g(\alpha)$ in Section 3.4.2, the function $h(i)$ is strictly increasing in the domain $\{i : -\bar{\alpha} \leq i \leq \infty\}$ for any nonnegative constant $\bar{\alpha}$ and S. Hence, a unique value of i satisfies the condition above, and this value can be approximated numerically by minimising the absolute value of $h(i)$.[2] Figure 3.2 illustrates the approximated value of i for the S&P 500 index options data. It is shown in Figure 3.2 that the value of i is considerably small for the entire sample period, and therefore it is reasonable to use linear approximation.

3.5. Empirical analysis

This section reports the results of the empirical analysis on the effectiveness of LE. The sensitivity function defined in Section 3.3 and the empirical methodology described in

[2]We can also define the minimum and maximum log-moneyness functions, i.e., $f(x)$ and $g(x)$, as exponential functions if we need to measure truncation while using strike price as the unit of measure more strictly.

Section 3.4 are employed to estimate the truncation sensitivity of the implied volatility, skewness, and kurtosis estimators of Bakshi et al. (2003), when they are used with and without LE on S&P 500 index options data.[3] Section 3.5.1 compares the implied moment estimation results with and without LE to demonstrate the effect of LE in outline. Section 3.5.2 compares the truncation sensitivity of the implied moment estimators with and without LE to investigate the effectiveness of LE. A description of the S&P 500 index options data used in this chapter is provided in Section 2.3.

3.5.1 Implied moment estimate with and without LE

Figure 3.3 illustrates the level of implied moment estimates during the sample period with and without the application of LE. A number of interesting points can be found from the figure. First, LE is shown to have a large influence on implied moment estimate when it is used with a high moment estimator. Determining the difference between Figures 3.3a and 3.3b visually is difficult, whereas the shape of the lines in Figure 3.3d is evidently different from that in Figure 3.3c, and the difference is even larger in scale between Figures 3.3e and 3.3f. Second, LE is shown to affect the implied skewness and kurtosis estimates significantly for a part of the sample period, whereas the effect is less significant for the other parts. Figures 3.3c–3.3f show that the implied skewness and kurtosis estimates are changed significantly after LE during the period from 2004 to 2007, whereas the change is less outstanding for the other periods. Finally, although a notable change in the average level and short-term dynamics of the implied moment estimates after LE can be observed, no significant change occurs in terms of the mid- and long-term trend of moment estimate fluctuation. If the noisy estimates are ignored in Figures 3.3c–3.3f, the lines in the corresponding pair of subfigures show a considerable similarity in shape.

3.5.2 Truncation sensitivity of implied moment estimate with and without LE

Table 3.1 reports the truncation sensitivity comparison result for the implied moment estimates with and without LE. The comparison is conducted by approximation of the nominal and percentage change in the estimates after an increase in strike price domain length by five, as explained in Section 3.4.3, followed by an investigation into whether a

[3]The maturities of 2 and 4-months are chosen because the daily options data can be obtained without any missing trading days only for the maturities of 2, 3, and 4 months after filtering.

significant difference exists in the mean of the absolute nominal and percentage changes before and after LE.

Panel A of Table 3.1 shows the comparison result for the absolute nominal change in estimate. Panel A depicts that the decrease in truncation sensitivity is statistically significant for all cases, except the implied kurtosis estimate for the maturity of two months.[4] Furthermore, Panel B of Table 3.1, which reports the comparison result for the absolute percentage change in estimate, indicates that the moment estimate becomes less sensitive to a small change in option price availability for all moments and maturities. With these results, LE can be conjectured to be effective and makes the estimates less sensitive to a change in option price availability for all three implied moment estimators of Bakshi et al. (2003). Another notable finding is that the truncation sensitivity of the implied volatility estimator is extremely low regardless of the application of LE. This finding suggests that the implied volatility estimator of Bakshi et al. (2003) is considerably robust to truncation when applied to S&P 500 index options data.

However, when we focus on the estimates on which LE has a relatively strong influence, a contradictory finding can be obtained. Figure 3.4 illustrates the approximated time-series dynamics of the nominal change in estimate after a change in strike price domain length by five. In Figures 3.4c–3.4f, most of the estimates that are highly sensitive to truncation without LE become even more sensitive with LE. Furthermore, the result is again similar in Figure 3.5, in which the size of the absolute percentage change is employed, so that the effect of the implied moment level is controlled. Given that the highly truncation-sensitive estimates are mostly located in the time period from 2004 to 2007, for which LE is shown to have the largest influence in Section 3.5.1, LE can be conjectured to have a reverse effect when the implied moment estimate is mostly determined by the option prices that are generated by LE. This is possible because if the estimate is mostly determined by the generated options, the change in the endpoint implied volatility level due to the change in the integration domain width will have a strong effect on the implied moment estimate.

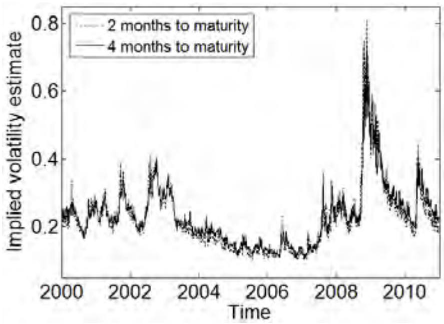
Overall, the empirical results suggest that LE is effective and makes the implied moment estimate less sensitive to truncation for all three estimators of Bakshi et al. (2003). However, LE might also have an adverse effect and make the estimate even more sensitive

[4] The difference is shown to be statistically significant regardless of most confidence intervals with and without LE being overlapped with each other, because this is a test on the difference between pairs, not means.

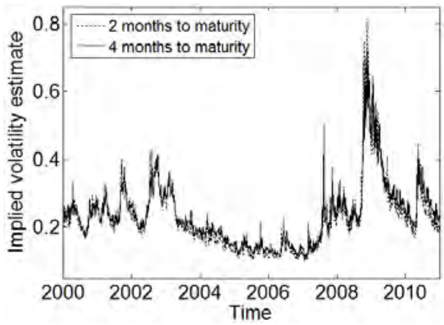
to truncation when the estimation relies on LE too heavily. Hence, a supplementary trun-

Figure 3.3: Implied moment estimate

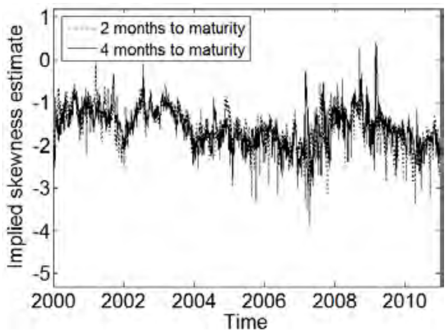
This figure illustrates the level of implied volatility, skewness, and kurtosis estimates with and without the application of LE. The S&P 500 index options dataset used spans a time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months that are extracted from the daily implied volatility surfaces. When LE is applied, the implied volatility level at the minimum and maximum strike prices are extrapolated up to the points at which the strike prices are $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level at day t.



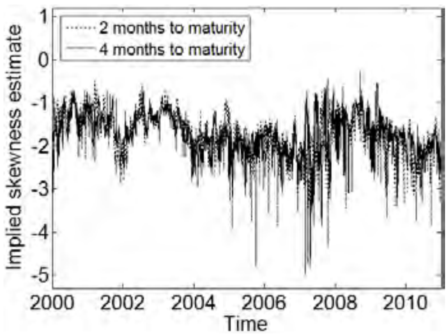
(a) Implied volatility estimate (without LE)



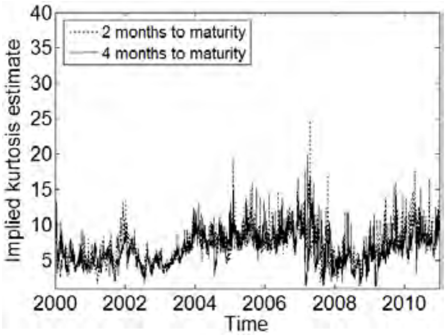
(b) Implied volatility estimate (with LE)



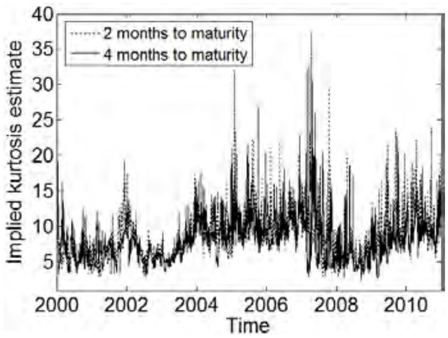
(c) Implied skewness estimate (without LE)



(d) Implied skewness estimate (with LE)



(e) Implied kurtosis estimate (without LE)



(f) Implied kurtosis estimate (with LE)

Table 3.1: Comparison of truncation sensitivity with and without LE

This table reports the truncation sensitivity comparison result for the implied volatility, skewness, and kurtosis estimators with and without LE. The comparison is conducted by applying the estimators on S&P 500 index options dataset, and then approximating the nominal and percentage change in the estimates after an increase in strike price domain length by five. The dataset spans an eleven year time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months, which are extracted from daily implied volatility surfaces. When LE is applied, implied volatility at minimum and maximum strike prices are extrapolated up to the points in which strike prices are $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-free index level at day t. The nominal change in estimate is approximated by first finding the value of increment i which minimises the absolute value of function $h(t, \tau, i)$, which is defined as

$$h(t, \tau, i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i)} - Se^{f(\bar{\alpha}(t,\tau)+i)}) - (S(t)e^{g(\bar{\alpha}(t,\tau))} - Se^{f(\bar{\alpha}(t,\tau))}) - 5,$$

where $f(\alpha, t, \tau) = c(t, \tau)\alpha$, $g(\alpha) = \alpha$, $c(t, \tau) = \ln[K_{\min}(t, \tau)/S(t)]/\ln[K_{\max}(t, \tau)/S(t)]$, $K_{\min}(t, \tau)$ and $K_{\max}(t, \tau)$ are the observed minimum and maximum strike prices for day t and maturity τ, respectively, and $\bar{\alpha}(t, \tau) = \ln[K_{\max}(t, \tau)/S(t)]$. Then, the nominal change for each (t, τ) is approximated as the truncation sensivity multiplied by i. Finally, the percentage change is calculated by dividing the nominal change by the corresponding implied moment estimate. The truncation sensitivity of implied volatility estimate is multiplied by 10,000 for better visibility, because the implied volatility estimate is found to be significantly insensitive to a change in option price availability when the estimator is applied to the dataset used in this chapter. ** and * denote statistical significance at the 1% and 5% levels, respectively.

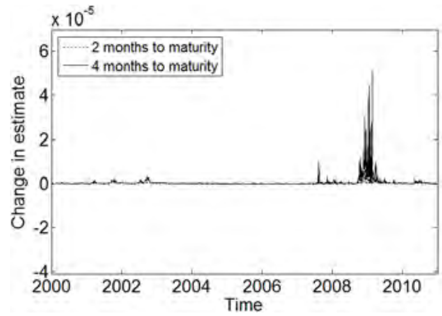
Estimate	Maturity	Application of LE	Mean	Standard deviation	Mean difference
Panel A. Absolute nominal change					
Implied volatility estimate (Sensitivity×10,000)	2 months	Without LE	0.0016	0.0051	0.0006**
		With LE	0.0010	0.0034	(5.57)
	4 months	Without LE	0.0062	0.0253	0.0015*
		With LE	0.0047	0.0292	(2.01)
Implied skewness estimate	2 months	Without LE	0.0093	0.0061	0.0005*
		With LE	0.0088	0.0089	(2.31)
	4 months	Without LE	0.0065	0.0048	0.0009**
		With LE	0.0056	0.0067	(5.60)
Implied kurtosis estimate	2 months	Without LE	0.0851	0.0594	−0.0016
		With LE	0.0867	0.0939	(−0.75)
	4 months	Without LE	0.0525	0.0400	0.0042**
		With LE	0.0483	0.0621	(2.95)
Panel B. Absolute percentage change					
Implied volatility estimate (Sensitivity×10,000)	2 months	Without LE	0.0043	0.0086	0.0016**
		With LE	0.0026	0.0065	(7.98)
	4 months	Without LE	0.0165	0.0464	0.0047**
		With LE	0.0118	0.0548	(3.41)
Implied skewness estimate	2 months	Without LE	0.0058	0.0034	0.0012**
		With LE	0.0046	0.0036	(12.19)
	4 months	Without LE	0.0043	0.0043	0.0013**
		With LE	0.0030	0.0030	(13.08)
Implied kurtosis estimate	2 months	Without LE	0.0112	0.0048	0.0028**
		With LE	0.0084	0.0059	(19.29)
	4 months	Without LE	0.0073	0.0033	0.0025**
		With LE	0.0048	0.0039	(25.47)

Figure 3.4: Change in estimate after a change in strike price domain length by five

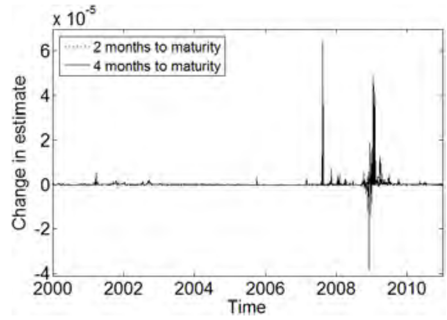
This figure reports the size of change in the implied volatility, skewness, and kurtosis estimates with and without LE, after an increase in strike price domain length by five. The S&P 500 index options dataset used here spans an eleven year time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months, which are extracted from daily implied volatility surfaces. When LE is applied, implied volatility at minimum and maximum strike prices are extrapolated up to the points in which strike prices are $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-free index level at day t. Approximation of size of change in estimate is done by first finding the value of increment i which minimises the absolute value of function $h(t, \tau, i)$, which is defined as

$$h(t, \tau, i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i)} - Se^{f(\bar{\alpha}(t,\tau)+i)}) - (S(t)e^{g(\bar{\alpha}(t,\tau))} - Se^{f(\bar{\alpha}(t,\tau))}) - 5,$$

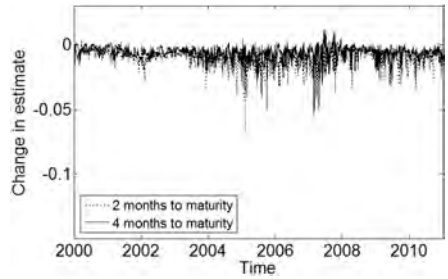
where $f(\alpha, t, \tau) = c(t, \tau)\alpha$, $g(\alpha) = \alpha$, $c(t, \tau) = \ln[K_{\min}(t, \tau)/S(t)]/\ln[K_{\max}(t, \tau)/S(t)]$, $K_{\min}(t, \tau)$ and $K_{\max}(t, \tau)$ are the observed minimum and maximum strike prices for time t and maturity τ, respectively, and $\bar{\alpha}(t, \tau) = \ln[K_{\max}(t, \tau)/S(t)]$. Then, the size of change is approximated as the truncation sensivity multiplied by i.



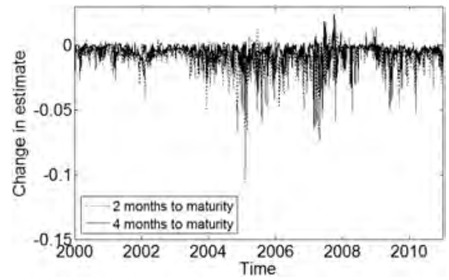
(a) Implied volatility estimate (without LE)



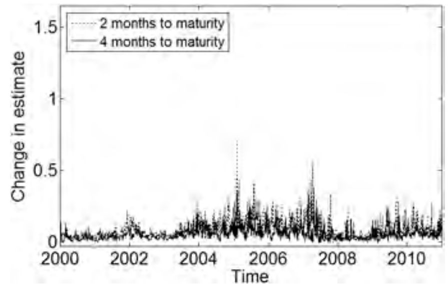
(b) Implied volatility estimate (with LE)



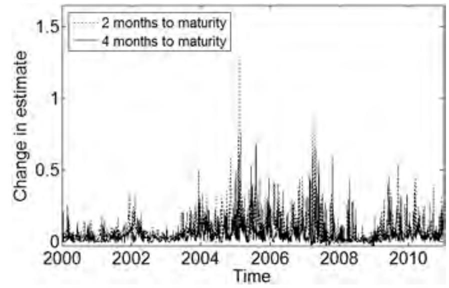
(c) Implied skewness estimate (without LE)



(d) Implied skewness estimate (with LE)



(e) Implied kurtosis estimate (without LE)



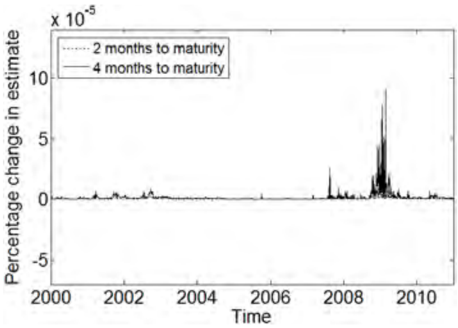
(f) Implied kurtosis estimate (with LE)

Figure 3.5: Percentage change in estimate after a change in strike price domain length by five

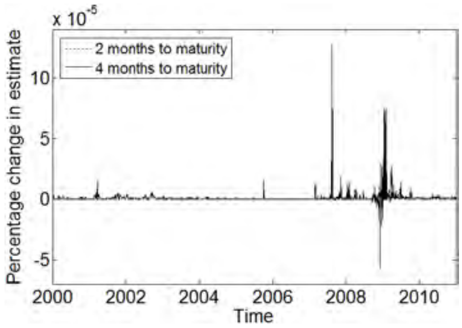
This figure reports the percentage change in the implied volatility, skewness, and kurtosis estimates with and without LE, after an increase in strike price domain length by five. The S&P 500 index options dataset used here spans an eleven year time period from January 2000 to December 2010. Option prices are estimated based on the implied volatility curve for the maturities of two and four months, which are extracted from daily implied volatility surfaces. When LE is applied, implied volatility at minimum and maximum strike prices are extrapolated up to the points in which strike prices are $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-free index level at day t. Rate of change in estimate is approximated by first finding the value of increment i which minimises the absolute value of function $h(t,\tau,i)$, which is defined as

$$h(t,\tau,i) = (S(t)e^{g(\bar{\alpha}(t,\tau)+i)} - Se^{f(\bar{\alpha}(t,\tau)+i)}) - (S(t)e^{g(\bar{\alpha}(t,\tau))} - Se^{f(\bar{\alpha}(t,\tau))}) - 5,$$

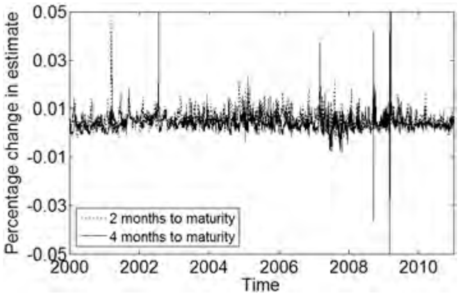
where $f(\alpha,t,\tau) = c(t,\tau)\alpha$, $g(\alpha) = \alpha$, $c(t,\tau) = \ln[K_{\min}(t,\tau)/S(t)]/\ln[K_{\max}(t,\tau)/S(t)]$, $K_{\min}(t,\tau)$ and $K_{\max}(t,\tau)$ are the observed minimum and maximum strike prices for time t and maturity τ, respectively, and $\bar{\alpha}(t,\tau) = \ln[K_{\max}(t,\tau)/S(t)]$. Next, the size of change is approximated as the truncation sensivity multiplied by i. Then, finally, the percentage change is calculated by dividing the size of change by the corresponding implied moment estimate.



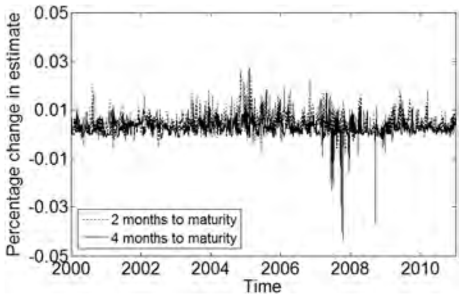
(a) Implied volatility estimate (without LE)



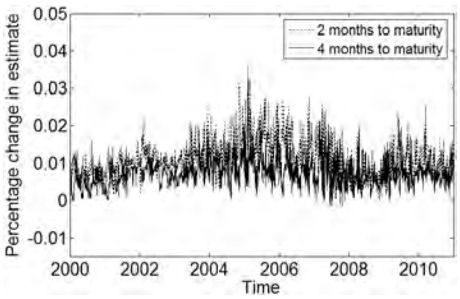
(b) Implied volatility estimate (with LE)



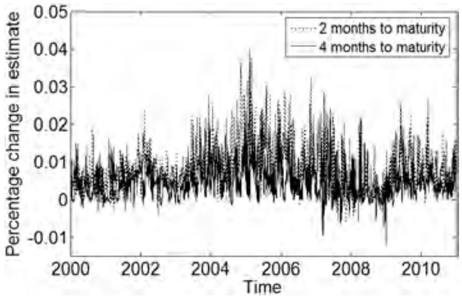
(c) Implied skewness estimate (without LE)



(d) Implied skewness estimate (with LE)



(e) Implied kurtosis estimate (without LE)



(f) Implied kurtosis estimate (with LE)

cation treatment may be required when LE is used in conjunction with estimators that are more closely related to the DOTM option prices and thus might rely heavily on LE. A simple supplementary treatment is to remove observations with extremely high truncation sensitivity so that the estimation is less affected by abrupt changes in the truncation error.

3.6. Conclusion

The non-parametric estimation of the implied RND has been a topic of recent interest, especially when the target of estimation is a moment of the density. DOTM option prices have vital information on the tail distribution but are only partially available in most options markets, so the missing DOTM option prices are required to be inferred from the option prices available when the implied moments are estimated non-parametrically. Sophisticated DOTM option price estimation methods may impair the model-freeness of the non-parametric implied moment estimators; therefore, LE has been a popular choice as a truncation treatment method owing to its simplicity and approximation-based approach. As the implied moment estimators of Bakshi et al. (2003) have become popular, LE has also drawn increased academic interest and has been frequently used in combination with the higher moment estimators. Nevertheless, less attention has been devoted to the issues of how effectively LE can reduce truncation error and whether LE can alleviate the truncation error of higher moment estimators.

This chapter addresses both of these issues and introduces an empirical methodology, i.e., measurement of truncation sensitivity, with which the effectiveness of LE can be assessed. If the truncation error becomes negligible when LE is applied, an estimate should not change significantly after a marginal change in option price availability. In other words, if an estimate changes significantly after including or excluding few options even when LE is applied, the truncation error can be conjectured to be not fully reduced by LE. Hence, the sensitivity of the implied moment estimate to a marginal change in option price availability can be used to assess how effectively LE reduces the truncation error. Basing on this idea, this chapter defines the truncation sensitivity functions for the implied volatility, skewness, and kurtosis estimators of Bakshi et al. (2003) with and without LE applied, and then employs these functions to approximate how the estimates will change after a small increase in the number of option prices available for S&P 500 index options market.

This chapter makes three contributions to the literature on model-free implied moment estimation. First, the methodology used in this chapter shows how a truncation treatment method for implied moment estimation can be assessed when the true value of implied moment is unknown so that the truncation error cannot be calculated. Because truncation sensitivity estimation does not require the true value of implied moments, the methodology in this paper can be employed for a set of European OTM options whose strike prices do not fully span the positive real line, and is therefore practical. Second, this chapter shows how the efficiency of LE varies according to the type of implied moment estimator in which LE is employed. Although LE is frequently used in conjunction with implied moment estimators other than the implied volatility estimator of Britten-Jones and Neuberger (2000) for which LE is first introduced, no studies validating this extended use of LE for different implied moment estimators have been conducted. This chapter fills this gap and empirically shows that LE is considerably effective when used in combination with the implied volatility estimator of Bakshi et al. (2003). However, the results also suggest that the remaining truncation error is not small enough to be regarded as negligible when LE is used with the implied skewness or kurtosis estimator of Bakshi et al. (2003). Finally, the truncation sensitivity function that is introduced in this chapter can also be used as an instrument for supplementary truncation treatment along with LE. Observations with a large truncation error can be detected by measurement of truncation sensitivity, so an additional data filter based on truncation sensitivity can be employed to reduce the effect of truncation on implied moment estimation.

Although this chapter points out an issue on the use of LE in combination with the implied skewness or kurtosis estimator of Bakshi et al. (2003), it does not invalidate LE or any argument in the work of Jiang and Tian (2005). On the contrary, the argument of Jiang and Tian (2005) that LE reduces the truncation error of implied volatility estimator is supported by this chapter because we show that such is also the case for the implied volatility estimator of Bakshi et al. (2003). In addition, LE also makes the implied skewness and kurtosis estimators less sensitive to truncation in most cases. Hence, this chapter still suggests the use of LE in conjunction with the implied moment estimators including those for higher moments, but only on the condition that the size of the remaining truncation error be properly controlled by supplementary methods when LE is employed for higher moment estimation. As suggested previously, a simple but effective supplementary method is to discard the observations whose truncation sensitivity is abnormally high, so that the estimation results are not distorted by abrupt changes in the truncation error.

Chapter 4

Integration domain symmetry and model-free implied skewness estimator

Chapter Summary

This chapter analyses the effectiveness of DSym, which is suggested by Dennis and Mayhew (2002) to minimise the truncation error of the implied skewness estimator of Bakshi et al. (2003), by examining the impact of domain asymmetry on the implied skewness estimator. This chapter shows that DSym effectively reduces the estimation bias if the symmetry of the integration domain is defined in terms of log-moneyness and the implied RND is symmetric. In addition, we reveal that the bias may not be reduced by DSym if any of the conditions is violated.

4.1. Introduction

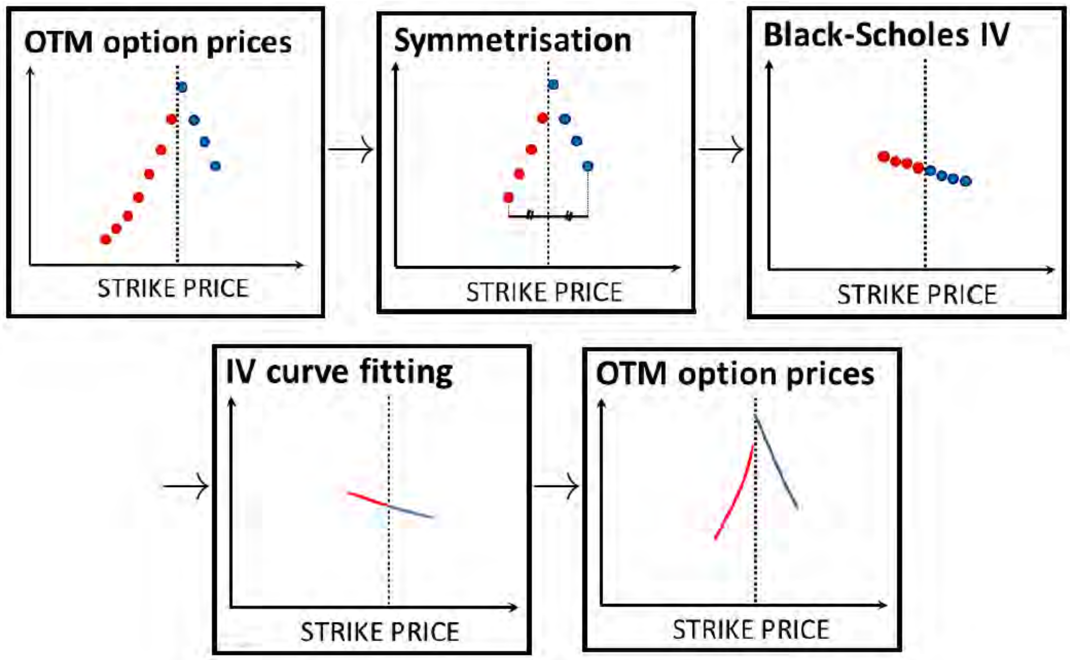
After Bakshi et al. (2003) have introduced a set of estimators with which the volatility, skewness, and kurtosis of the implied RND can be measured nonparametrically, the studies in the field of financial derivatives have immediately adopted the estimators and started using them to extract information from OTM option prices. The study of Dennis and Mayhew (2002), who employ the estimators even before the work of Bakshi et al. (2003) has been published, is one of those pioneering studies that realize the usefulness of the model-free implied moment estimators early and promote the use of those estimators. In addition, Dennis and Mayhew (2002) also conduct a brief analysis on the implied skewness estimator using a set of model-based generated OTM option prices, and suggest how option price data should be processed before being used for implied skewness estimation.

Integration domain asymmetry is one of the implementation issues Dennis and Mayhew (2002) point out in their analysis of the implied skewness estimator. When there exists a truncation, the integration domain can be asymmetric, i.e., the distance of the minimum and maximum values of truncated strike price domain from the underlying price can be different from each other. Dennis and Mayhew (2002) argue that since the fair value estimates of the moment-related contracts, i.e., V, W, and X, depend on the difference between the weighted average of OTM calls and OTM puts, having more call option price observations than put option price observations can introduce bias. To address this issue, Dennis and Mayhew (2002) suggest DSym, i.e., an additional discardment of observed option quotes which in order to equalise the distance of the minimum and maximum strike prices of integration domain from the underlying price. Figure 4.1 illustrates how Dsym can be conducted in an empirical analysis.

Although the test results in Dennis and Mayhew (2002) provide a deep insight into the implied skewness estimator and make the idea of DSym considerably persuasive, there is still a need to further examine the effectiveness of DSym given that the implied skewness estimator has been popularly used and DSym itself has been adopted by recent studies (e.g., Conrad et al., 2013). Specifically, there are two questions regarding DSym that can be of interest. First, why is the integration domain extremely asymmetric when there is no truncation? It can be easily found that when integration domain is symmetric in terms of strike price as in Dennis and Mayhew (2002), it must be possible to define the integration

Figure 4.1: Process of domain symmetrisation

This figure demonstrates how Dsym can be conducted in an empirical analysis. Dennis and Mayhew (2002) conduct Dsym by equalising the number of OTM calls and puts. After Dsym, the symmetrised option prices can be converted to Black-Scholes implied volatility values, from which an implied volatility curve is estimated to reduce the issue of strike price discreteness.



domain as $[S - c, S + c]$, where S is the underlying asset price and $c \leq S$ is a positive real constant. In other words, the domain can be regarded as asymmetric if the distance is different for the minimum and maximum values of the integration domain from the underlying asset price S. The issue is that when there is no truncation so that the integration domain becomes $(0, \infty)$, then the distance becomes S for the minimum value but infinite for the maximum value. This is inconsistent with the main idea of DSym and, therefore, there is a need to investigate where this inconsistency comes from. Second, does DSym also minimise the estimation bias even when the implied RND is skewed? When evaluating the effectiveness of DSym, Dennis and Mayhew (2002) assume that the underlying price follows a Black-Scholes constant volatility process so that the implied RND is normal. However, it is well known that the Black-Scholes constant volatility assumption is not applicable for most options markets. Furthermore, if the implied RND is skewed so that option prices tend to be higher at one side, option price unavailability at each side may have a different size of impact on the implied skewness estimator even when the intergration domain is symmetric. Hence, there is a need to test the efficiency of DSym while assuming the the implied RND is skewed.

This chapter addresses the questions above and provides a deeper understanding of the relationship between the model-free implied skewness estimator and the asymmetry of the integration domain. Specifically, we first suggest the preconditions that is required for DSym to alleviate the truncation error effectively, and then show how the effectiveness of DSym changes when those preconditions are violated.

The analysis in this chapter reveals three interesting findings. First, when the implied RND is symmetric, DSym becomes more effective when the integration domain is symmetric in terms of log-moneyness. This is different from the conclusion of Dennis and Mayhew (2002) who define integration domain symmetry in terms of strike price. This study shows that this difference is a key to answer the first question above, i.e., the question of why the integration domain is extremely asymmetric when there is no truncation. Second, the truncation error is not effectively reduced with DSym if the true skewness is non-zero, and the effectiveness of DSym depends on the true level of implied skewness. Specifically, the error is reduced if the integration domain is biased to the OTM put side when the true implied skewness is negative, whereas the error decreases if the domain is biased to the OTM call side when the true implied skewness is positive. Finally, the size of the truncation error also depends on the width of the integration domain, even when the degree of domain asymmetry is fixed. The error tends to increase as the integration domain becomes smaller. Overall, the results suggest that DSym may increase the size of the truncation error if the true skewness is non-zero or the integration domain is significantly asymmetric so that DSym will result in a considerable decrease in the width of integration domain.

The rest of this chapter is organised as follows. Section 4.2 explains when DSym can be effective. Section 4.3 investigates the impact of the integration domain asymmetry on implied skewness estimator using model-based generated option prices. Section 4.4 reports the results of the empirical analysis. Section 4.5 concludes the chapter.

4.2. Under which condition does DSym become effective?

For the skewness of the truncated density g in Section 2.2.3, the following example can explain under what condition DSym can be effective. Suppose that (1) the minimum and maximum log-moneyness values have the same magnitude, i.e., $\ln(K_{\min}(t,\tau)/S(t)) = -\ln(K_{\max}(t,\tau)/S(t))$, and (2) $f(x) = f(-x)$ for the implied RND function f and any real x. Then $g(x) = -g(x)$ for any real x, and therefore the skewness of both f and g are zero. Hence, if the impact of truncation on the expected value is negligible, the estimation bias of implied skewness estimator is minimised when these two conditions are satisfied.

The example above implies what the preconditions for DSym are. Firstly, integration domain symmetry needs to be defined in terms of log-moneyness, not strike price. Specifically, an integration domain $[K_{\min}(t,\tau), K_{\max}(t,\tau)]$ should be regarded as symmetric if $\ln(K_{\min}(t,\tau)/S(t)) = -\ln(K_{\max}(t,\tau)/S(t))$. This is slightly different to the definition of symmetry in Dennis and Mayhew (2002), where a domain $[K_{\min}(t,\tau), K_{\max}(t,\tau)]$ is regarded as symmetric if $K_{\min}(t,\tau) - S(t) = -(K_{\max}(t,\tau) - S(t))$. Secondly, the true RND must be symmetric. This is the case in Dennis and Mayhew (2002), where they employ the Black-Scholes constant volatility assumption when the effectiveness of DSym is tested. Finally, the probability density defined by g must be risk-neutral to obtain the unbiased skewness estimate. On the other hand, if any of the first two conditions are not satisfied, then the skewness of the density defined by g becomes more likely to be asymmetric, and therefore it is hard to ensure that DSym successfully reduces the estimation bias.

The point that integration domain symmetry needs to be defined in terms of log-moneyness to ensure the effectiveness of DSym suggests why the truncated strike price domains are heavily asymmetric when the integration domain is symmetric. In fact, the truncated domains can also be regarded as symmetric in terms of log-moneyness when the integration domain is symmetric in terms of log-moneyness, because the truncated domains are then $(-\infty, \ln(K_{\min}(t,\tau)/S(t)))$ and $(\ln(K_{\max}(t,\tau)/S(t)), \infty)$. Although they are still heavily asymmetric in terms of strike price, it is less relevant because the implied skewness estimator measures the implied skewness of log-return density, not the underlying price density.

Section 4.3 shows that the first two conditions above are in fact closely related to the effectiveness of DS. First, we show that if the Black-Scholes constant volatility assumption is employed so that implied RND is symmetric, DSym reduces the estimation bias better when the integration domain symmetry is defined in terms of log-moneyness, not strike price. Next, we demonstrate that if implied RND becomes asymmetric, the effectiveness of DSym becomes less significant. This result is supported by an empirical analysis of S&P 500 index options market, whose implied RND is reported to be negatively skewed by several studies.

4.3. Effectiveness of DSym on model option prices

This section shows how the implications in Section 4.2 are reflected in the implied skewness estimate, using two sets of model OTM option prices. Specifically, we demonstrate that (1) domain symmetry in terms of log-moneyness, not strike price, is more effecitve in reducing estimation bias when implied RND is symmetric, and (2) DSym may not be effective when the implied RND is asymmetric. Section 4.3.1 investigates under which definition of integration domain symmetrty DSym reduces the estimation bias more effectively, when the implied RND is symmetric. Section 4.3.2 examines whether DSym reduces estimation bias effectively when the implied RND is asymmetric. A description about the model-based generated option prices is provided in Section 2.4.1.

4.3.1 Definition of integration domain symmetry

Section 4.2 suggests that if the implied RND is symmetric, then the differnce between the skewness of implied RND and that of the truncated density is minimised when the integration domain is symmetric in terms of log-moneyness, i.e., $\ln(K_{\min}(t,\tau)/S(t)) = -\ln(K_{\max}(t,\tau)/S(t))$. To further investigate how this condition affects the implied skewness estimator when the implied RND is symmetric, we examine if DSym with this new definition of domain symmetry reduces the estimation bias better than the one with the original definition of domain symmetry in Dennis and Mayhew (2002), i.e., the state in which $K_{\min}(t,\tau) - S(t) = -(K_{\max}(t,\tau) - S(t))$. Based on the two different definitions of domain symmetry, we vary the level of domain asymmetry using an asymmtery coefficient c, and then examine the relationship between the level of asymmetry and the size of estimation bias. When the integration domain symmetry is defined in terms of strike price difference as in Dennis and Mayhew (2002), for a given domain half-width W which satistfies $K_{\max}(t,\tau) - K_{\min}(t,\tau) = 2W$, the level of domain asymmetry is controlled by setting $K_{\min}(t,\tau)$ and $K_{\max}(t,\tau)$ as $S-(1-c)W$ and $S+(1+c)W$, respectively. With this specification, the ratio of strike price distances from the underlying price to the minimum and maximum strike prices becomes

$$S(t) - K_{\min}(t,\tau) : K_{\max}(t,\tau) - S(t) = 1 - c : 1 + c.$$

On the other hand, when the domain symmetry is defined in terms of log-moneyness, $K_{\min}(t,\tau)$ and $K_{\max}(t,\tau)$ are set in a way that the following conditions hold:

$$\begin{cases} (1+c)\ln(S(t)/K_{\min}(t,\tau)) = (1-c)\ln(K_{\max}(t,\tau)/S(t)); \text{ and} \\ K_{\max}(t,\tau) - K_{\min}(t,\tau) = 2W. \end{cases}$$

Here the ratio of log-moneyness distances becomes

$$\ln(S(t)/K_{\min}(t,\tau)) : \ln(K_{\max}(t,\tau)/S(t)) = 1 - c : 1 + c.$$

In both cases, the integration domain is regarded as symmetric when $c = 0$, biased to the OTM put side when $c < 0$, and biased to the OTM call side when $c > 0$. BS model option prices in Chapter 2 are used to set the implied RND symmetric.

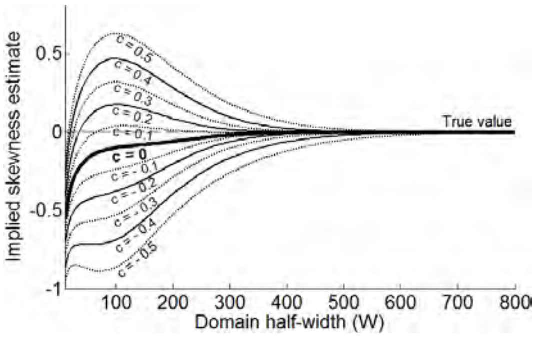
Figure 4.2 demonstrates how the implied skewness estimate varies when the level of integration domain asymmetry changes. The figure reveals that when the implied RND is symmetric, the estimation bias tends to be larger when domain is more asymmetric. This is consistent with Dennis and Mayhew (2002). However, in Figure 4.2a in which the domain symmetry is defined in terms of strike price, it can be found that the skewness estimate converges to the true value most quickly as the width of the integration domain increases when the integration domain is slightly biased to the OTM call side. On the other hand, in Figure 4.2b in which the domain symmetry is defined in terms of log-moneyness, estimate is shown to converge to true value most quickly when domain is symmetric. This supports the implication of Section 4.2 that DSym is more effective when the domain symmetry is defined in terms of log-moneyness, on the condition that the implied RND is symmetric.

Figure 4.2: Definition of domain symmetry and implied skewness estimate

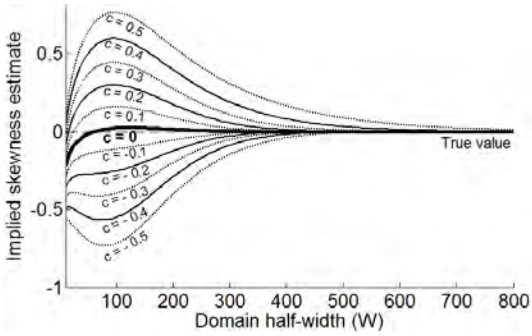
This figure illustrates the relationship between the domain asymmetry level and the implied skewness estimate. In Figure 4.2a, for a fixed domain half-width W, strike price domain is set to be $[S-(1-c)W, S+(1+c)W]$, where $S = 1178.3$ is the dividend-adjusted underlying price and c is the asymmetry coefficient. In Figure 4.2b, on the other hand, strike price domain is set to satisfy the following conditions for each W:

$$(1+c)\ln(S/K_{\min}) = (1-c)\ln(K_{\max}/S); \quad \text{and} \quad K_{\max} - K_{\min} = 2W,$$

where $K_{\min}$ and $K_{\max}$ are the minimum and maximum strike prices, respectively. The BS model option prices in Chapter 2 are employed. The strike price interval is set as 0.1.



(a) BS (strike price based)



(b) BS (log-moneyness based)

4.3.2 Effectiveness of DSym when the implied RND is skewed

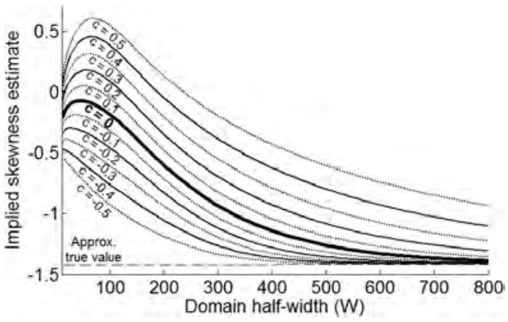
As mentioned in Section 4.2, one of the preconditions for DSym to effectively reduce the truncation error is that the implied RND should be symmetric. In this subsection, we investigate whether DSym effectively reduces the truncation error even when the implied RND is skewed. Given the result in Section 4.3.1, the symmetry of the integration domain is defined in terms of log-moneyness, and the level of domain asymmetry is again controlled using an asymmtery coefficient c. Figure 4.3 illustrates the level of implied skewness estimate when the SVJ model option prices that are used in Chapter 2 are employed so that the implied RND is skewed. Overall, the figure suggests that the truncation error is not effectively reduced by DSym when the implied RND is asymmetric. In Figure 4.3a in which the skewness of the implied RND is set to be negative, it can be found that the truncation error tends to be smaller when the integration domain is more biased to the OTM put side. On the other hand, in Figure 4.3b in which the sign of mean jump size μ and correlation coefficient ρ in the SVJ model in Chapter 2 are switched to be positive so that the implied RND is also positively skewed, it is shown that the truncation error tends to decrease as the integration domain becomes more biased to the OTM call side.

Figure 4.3: Asymmetric implied RND, domain symmetry, and implied skewness estimate

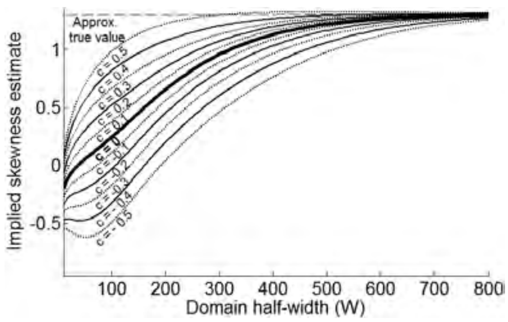
This figure illustrates the relationship between the domain asymmetry level and the implied skewness estimate. The strike price domain is set to satisfy the following conditions for each W:

$$(1+c)\ln(S/K_{\min}) = (1-c)\ln(K_{\max}/S); \quad \text{and} \quad K_{\max} - K_{\min} = 2W,$$

where $K_{\min}$ and $K_{\max}$ are the minimum and maximum strike prices, respectively. The SVJ model option prices in Chapter 2 are employed, while the sign of values for parameters μ and ρ are switched to be positive for Figure 4.3b in order to set the implied skewness positive. The straight line in each subfigure indicates the approximated true level of implied moment, which is obtained by an estimation using options for strike price domain $[3/S, 3S]$, where $S = 1178.3$ is the dividend-adjusted underlying price. respectively. The strike price interval is set as 0.1.



(a) SVJ (log-moneyness based)



(b) SVJ (sign switched, log-moneyness based)

In combination with the results in Section 4.3.1, the results in this subsection suggest that the truncation error of the implied skewness estimator is effectively reduced by DSym on the condition that domain symmetry is defined in terms of log-moneyness and implied RND is symmetric, which is consistent with Section 4.2.

Another notable finding in Figure 4.3 is that the truncation error is determined not only by the level of integration domain asymmetry but also by the with of the integration domain. This finding is meaningful because DSym reduces the width of the integration domain while making the domain symmetric. Hence, it is difficult to conclude that DSym reduces the estimation bias even when the truncation error of the implied skewness estimator is minimised when the implied domain is symmetric. Namely, DSym may increase the size of the truncation error when the integration domain is severely asymmetric so that DSym results in a large reduction in the width of the integration domain.

4.4. Empirical analysis

Section 4.3 shows that DSym reduces the truncation error of the implied skewness estimator effectively on the condition that the integration domain is symmetric in terms of log-moneyness, and the implied RND is symmetric. In this section, we conduct a set of empirical analyses on S&P 500 index options dataset to obtain a deeper insight into the relationship between the integration domain asymmetry and the truncation error of the implied skewness estimator. Specifically, we examine how the level of integration domain asymmetry is related to the truncation error of the implied skewness estimator via a regression analysis. Section 4.4.1 demonstrates how the option prices are reconstructed from the dataset. Section 4.4.2 describes how the size of the truncation error is approximated for the S&P 500 index options dataset, from which the true value of implied skewness cannot be obtained. Section 4.4.3 examines the relationship between integration domain symmetry and the size of estimation bias. A description of the S&P 500 index options data used in this chapter is provided in Section 2.3.

4.4.1 Generation of implied volatility surface

Following Jiang and Tian (2005), daily implied volatility surfaces are generated to fix the maturities in order to mitigate the telescoping problem that is pointed out by Christensen et al. (2002).[1] To generate an implied volatility surface, all daily Black-Scholes implied volatility observations are first collected for different strike prices and maturities. Next, a bicubic spline funcation is fitted against the observations. When the implied skewness is estimated, implied volatility curves for two maturities, i.e., two and four months, are extracted from the daily surface. This is done by generating implied volatility observations between the minimum and maximum strike prices for the maturities with a strike price interval of 0.1, using the same bicubic spline function. If the minimum and maximum strike prices are not observable for a maturity, they are approximated by linearly interpolating the corresponding endpoint strike prices for the two nearest maturities for which the endpoints are observable. Finally, the implied volatility observations are converted to OTM option prices and then used for the implied skewness estimation.

[1] Christensen et al. (2002) point out that given the fixed maturity dates, time to maturity for an option is telescoping, i.e., decreasing over time, while making the time period between present date and maturity date overlapping for option samples in different days but with the same maturity date.

4.4.2 Approximation of truncation error

Given that the true level of implied skewness is unknown for the S&P 500 index options data, there is a need to choose a proxy variable with which the size of the truncation error can be approximated. One way to obtain such a proxy is to apply an alternative truncation error reduction method, and then measure the size of change in the implied skewness estimate that is occured following the application of the method. In this chapter, LE is employed as the alternative truncation error reduction method. We define the absolute percentage change (APC) as

$$\mathrm{APC} = \left| \frac{(\text{Estimate after LE}) - (\text{Estimate before LE})}{(\text{Estimate before LE})} \right|, \tag{4.1}$$

and use this variable to approximate the size of the truncation error for the rest of this section. When the LE is applied, the endpoint level of implied volatility curve is extrapolated up to the points where the strike prices are $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level on day t. Strike price interval between the observations generated by LE is set as 0.1. Table 4.1 reports some preliminary statistics of APC.

4.4.3 Integration domain asymmetry and truncation errors

This subsection examines the relationship between the integration domain symmetry and the truncation error of the implied skewness estimator via a regression analysis. The following two measures are defined to gauge the level of integration domain asymmetry:

$$(\text{Log-moneyness width difference ratio}) = \frac{\ln\left(K_{\max}(t,\tau)/S(t)\right) - |\ln\left(K_{\min}(t,\tau)/S(t)\right)|}{\ln\left(K_{\max}(t,\tau)/S(t)\right) + |\ln\left(K_{\min}(t,\tau)/S(t)\right)|}, \tag{4.2}$$

$$(\text{Log-moneyness width log-ratio}) = \ln\left(\frac{\ln(K_{\max}(t,\tau)/S(t))}{|\ln(K_{\min}(t,\tau)/S(t))|}\right). \tag{4.3}$$

Both of the measures above are based on the definition of domain symmetry in terms of log-moneyness. In addition, the following variable is also considered as an independent variable to control the impact of domain width on the size of estimation bias:

$$(\text{Log-moneyness width}) = \ln\left(K_{\max}(t,\tau)/S(t)\right) + |\ln\left(K_{\min}(t,\tau)/S(t)\right)|$$

Table 4.1: Summary statistics of truncation error proxy variable

This table presents a set of summary statistics of absolute percentage change (APC) in implied skewness estimate after LE, i.e.,

$$\text{APC} = \left| \frac{(\text{Estimate after LE}) - (\text{Estimate before LE})}{(\text{Estimate before LE})} \right|,$$

which is used as a proxy for truncation error in Section 4.4 Since this variable can have an abnormally high value for implied skewness estimate if the estimate originally has a near-zero value and its sign is switched after LE, we discard daily observations with absolute percentage change of implied skewness estimate larger than one thousand percent. There are three such observations in our sample, and 2,750 daily observations remain after this additional filtration.

Time period	Moment	Mean	Median	Std. dev.	5^{th} pct.	95^{th} pct.	N
Entire sample period	2 months	0.0990	0.0770	0.1294	0.0112	0.2460	2,750
	4 months	0.1338	0.0790	0.3244	0.0112	0.3371	2,750
2000–2003	2 months	0.1386	0.0992	0.1917	0.0170	0.3344	997
	4 months	0.1634	0.1238	0.1878	0.0150	0.3795	997
2004–2007	2 months	0.0732	0.0676	0.0570	0.0069	0.1563	999
	4 months	0.0780	0.0617	0.1436	0.0079	0.1540	999
2008–2010	2 months	0.0806	0.0690	0.0705	0.0114	0.1699	754
	4 months	0.1685	0.0736	0.5511	0.0172	0.3828	754

Figure 4.4 presents the histogram of the daily values of asymmetry measures for the maturity of two months.[2] The figure reveals that the values are almost always negative regardless of the measure employed, and most of the positive values are near zero. This suggests that the integration domain is biased to the OTM put side in most daily observations. Hence, it can be conjectured that if symmetry leads to a smaller truncation error, there should be a negative relationship between the measured value and the truncation error, because an increase in measured value does almost always mean a less negative value (which is closer to zero), rather than a more positive value (which is further away from zero).

[2]The histrograms for the maturity of four months are very similar to the one for the maturity of two months, and therefore omitted.

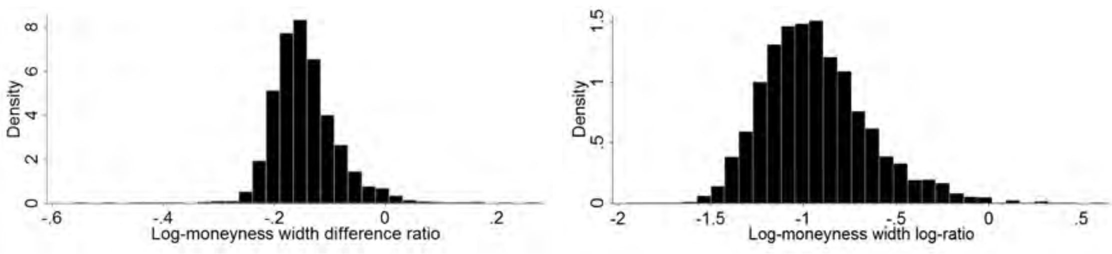
Figure 4.4: Sample distribution of the integration domain asymmetry level

This figure presents the sample distribution of the integration domain asymmetry level for the time to maturity of two months. The two asymmetry level measures used in this figure are defined as follows:

$$\text{(Log-moneyness width difference ratio)} = \frac{\ln\left(K_{\max}(t,\tau)/S(t)\right) - |\ln\left(K_{\min}(t,\tau)/S(t)\right)|}{\ln\left(K_{\max}(t,\tau)/S(t)\right) + |\ln\left(K_{\min}(t,\tau)/S(t)\right)|},$$

$$\text{(Log-moneyness width log-ratio)} = \ln\left(\frac{\ln(K_{\max}(t,\tau)/S(t))}{|\ln(K_{\min}(t,\tau)/S(t))|}\right),$$

where $K_{\min}(t,\tau)$ and $K_{\max}(t,\tau)$ are the minimum and maximum strike prices at time t and time to maturity τ, respectively, and $S(t)$ is the dividend-free S&P 500 index level at time t.



(a) Log-moneyness width difference ratio (b) Log-moneyness width log-ratio

Table 4.2 reports the regression result. The table shows a significantly positive relationship between the level of integration domain asymmetry and the size of the truncation error, regardless of which domain asymmetry measure is employed or whether the impact of integration domain width is considered. The positive relationship suggests that the truncation error of the implied skewness estimator tends to be smaller when the integration domain is more biased to the OTM put side, i.e., the truncation error tends to increase as the integration domain becomes more symmetric. This result is consistent with Section 4.3.2, in which the estimation bias is smaller when the integration domain is biased to the OTM put side. Hence, the result in Table 4.2 again supports the idea that DSym may not effectively reduce the truncation error when the implied RND is negatively skewed.[3]

[3] It is difficult to explain why the marginal effects of log-moneyness width on truncation error increase in twofold moving from two-month to four-month contracts, because 1) it is hard to tell whether this is due to the properties of truncation error or the properties of the proxy variable, APC, that are independent from truncation error, and 2) it is hard to tell whether this is due to the larger truncation error for longer maturities or the larger truncation sensitivity for longer maturities.

Table 4.2: Relationship between domain asymmetry and truncation error

This table reports the regression results of truncation error proxy variable, i.e., APC, on the level of integration domain asymmetry that is measured in two different ways. The two domain asymmetry level measures, i.e, log-moneyness width difference and log-moneyness width log-ratio, are defined as follows:

$$\text{(Log-moneyness width difference ratio)} = \frac{\ln(K_{\max}(t,\tau)/S(t)) - |\ln(K_{\min}(t,\tau)/S(t))|}{\ln(K_{\max}(t,\tau)/S(t)) + |\ln(K_{\min}(t,\tau)/S(t))|},$$
$$\text{(Log-moneyness width log-ratio)} = \ln\left(\frac{\ln(K_{\max}(t,\tau)/S(t))}{|\ln(K_{\min}(t,\tau)/S(t))|}\right).$$

Log-moneyness width, which is defined as

$$\text{(Log-moneyness width)} = \ln(K_{\max}(t,\tau)/S(t)) + |\ln(K_{\min}(t,\tau)/S(t))|$$

is considered as an independent variable to control the impact of the integration domain width on the truncation error size. ** and * denote statistical significance at the 1% and 5% levels, respectively.

	2 months				4 months			
	[1]	[2]	[3]	[4]	[1]	[2]	[3]	[4]
Log-moneyness width difference ratio	1.2092**		1.3153**		2.1813**		2.3106**	
	(35.09)		(37.44)		(36.03)		(38.23)	
Log-moneyness width log-ratio		0.2487**		0.2487**		0.4722**		0.4719**
		(35.33)		(35.33)		(31.07)		(30.88)
Log-moneyness width			-0.1320^{**}	0.0017			-0.2669^{**}	-0.0050
			(-10.86)	(0.14)			(-10.95)	(-0.20)
Constant	0.2728**	0.3303**	0.3461**	0.3295**	0.3935**	0.5456**	0.5707**	0.5484**
	(50.88)	(48.16)	(40.49)	(38.52)	(44.57)	(38.19)	(31.11)	(27.24)
Adj. R^2	0.3092	0.3123	0.3374	0.3120	0.3206	0.2596	0.3487	0.2594
N	2,750	2,750	2,750	2,750	2,750	2,750	2,750	2,750

4.5. Conclusion

One of the consequences of truncation is that the integration domain can be asymmetric, and the asymmetry may affect the implied skewness estimator. To address this issue, Dennis and Mayhew (2002) suggest DSym, i.e, a further reduction of integration domain that makes the domain symmetric. This chapter analyses the effectiveness of DSym by investigating whether and, if so, under what condition DSym minimises the truncation error of the implied skewness estimator. Specifically, we examine how integration domain asymmetry affects the implied skewness estimator using generated and observed option prices data.

The main findings of this chapter are as follows. First, DSym alleviates the estimation bias of implied skewness estimate effectively on the condition that the symmetry of integration domain is defined in terms of log-moneyness and the implied RND is symmetric. On the contrary, DSym may become less effective even when the implied RND is symmetric if the symmetry of the integration domain is defined in terms of strike price as in Dennis and Mayhew (2002). Second, the truncation error of the implied skewness estimator may not be reduced by DSym if the implied RND is asymmetric. Especially, the results in this paper imply that the bias is smaller when integration domain is more biased to OTM put side if the implied skewness is negative, and when the domain is more biased to OTM call side if the implied skewness is positive. Finally, the width of the integration domain is also closely related to the size of the truncation error, which implies that DSym may also increase the size of the truncation error by reducing the width of the integration domain.

Overall, this chapter suggests that the effectiveness of DSym relies on some preconditions and can be insignificant or even negative if any of the preconditions is violated. This implies that DSym should be employed carefully. Especially, an alternative method may be required if the implied RND is supposed to be significantly skewed, or the integration domain is extremely asymmetric so that applying DSym will result in a significant reduction in the width of the integration domain.

Chapter 5

Controlling the impact of truncation on model-free implied moment estimator

Chapter Summary

This chapter introduces DStab which makes the truncation error of the implied moment estimators of Bakshi et al. (2003) less volatile cross-sectionally and over time. An empirical analysis of the S&P 500 index options data suggests that the variance of the truncation error decreases when DStab is employed while the mean increases, whereas both the mean and variance increase when DSym is employed instead. In addition, DStab is shown to be less effective when a domain (a)symmetry measure is not based on log-moneyness or not adjusted for the market volatility, which suggests that the both factors are essential to control the level of truncation effectively.

5.1. Introduction

Chapters 3 and 4 show that it is difficult to eliminate the truncation error of the implied moment estimators. Chapter 3 suggests that the truncation error of the implied skewness and kurtosis estimators tends to be significant and varying over time even when LE is employed. Given that the S&P 500 index options market, which is one of the most liquid markets, is considered in Chapter 3, the size of the truncation error that survives LE can be even larger in the less liquid options markets, e.g., individual equity options market. In addition, Chapter 4 shows that DSym can be ineffective if the true level of the implied skewness is non-zero. Specifically, Chapter 4 reveals that the truncation error becomes smaller if the integration domain is more biased to the side where the OTM option prices tend to be more expensive than the other side, given the width of the integration domain fixed. Since many previous studies report that the implied skewness tends to be significantly negative in many options markets, it is likely that DSym in fact increases the size of the truncation error in most cases.

Given these results, it may be of interest to think about a new truncation error treatment method. However, given the limited availability of option prices and the existing methods that are equipped with their own reasonable logic, it is not easy to devise a new method that surpasses all the existing ones. Hence, this chapter first suggests to change the goal of the truncation error treatment before proposing a new method. It should be noted that the implied moment estimators of Bakshi et al. (2003) are employed in recent studies mostly to make a cross-sectional comparison of the implied moments across the options on different underlying assets, or to track the time-series dynamics of an implied moment, rather than measuring the level of the implied moments with the best precision. If this is the case, as argued by Dennis and Mayhew (2002), the 'de facto' effect of truncation on empirical analysis can be minimised if the size of truncation error is consistent across the entire observations. Hence, minimising the volatility of truncation error can be an alternative to minimising the mean if it is difficult to achieve the latter and if succeeding in the former will make the truncation error acceptable.

The fact that truncation error is due to truncation implies that the size of truncation error is related to the level of truncation, and therefore the former can be controlled by manipulating the latter. DSym indeed takes this idea and tries to reduce truncation error by further discarding OTM option prices. However, the issue of DSym is that the relationship between the level of truncation and the size of truncation error is not modelled accurately. Hence, in order to improve the effectiveness of DSym and make the truncation error less volatile, the relationship between the two factors should be modelled more accurately so that the size of the truncation error can be stabilised by controlling the level of truncation correspondingly.

Chapter 4 suggests that it is not the strike price but the log-moneyness of the endpoints of integration domain that should be considered to interpret the impact of truncation on the implied moment estimators. In addition, this chapter shows that the level of implied volatility also needs to be considered to explain the relationship between the truncation level and the truncation error size. Based on these findings, this chapter then reveals that if the level of truncation is defined using the endpoint log-moneyness that is adjusted by the level of implied volatility, a strong relationship between the level of truncation and the size of truncation error can be found. Based on these findings, this chapter finally suggests a new truncation treatment method, DStab, that makes the truncation error less volatile cross-sectionally and over time. The variance comparison test on the proxy variable for the size of the truncation error shows that the truncation error volatility reduction effect of DStab is statistically significant. In addition, the test results also suggest that the volatility of truncation error cannot be reduced consistently if the specification of DStab is modified or DSym is employed instead.

The rest of this chapter is constructed as follows. Section 5.2 analyses the relationship between the level of truncation and the size of truncation error. Section 5.3 introduces DStab and explains how it is conducted. Section 5.4 summarises the result of empirical analysis. Section 5.5 discusses the main findings and concludes.

5.2. Relationship between truncation level and truncation error size

This section analyses the relationship between the level of truncation and the size of truncation error. Chapter 4 shows that the impact of truncation on model-free implied moment estimator is closely related to the log-moneyness of the minimum and maximum strike price of the integration domain, and that both the width and asymmetry level of integration domain affect the moment estimate. In addition to these findings, Section 5.2.1 shows that the size of truncation error is also related to the level of implied volatility, and proposes the implied volatility adjusted log-moneyness (IVAL) as a new measure of truncation level. Section 5.2.2 suggests that the relationship between the truncation level and the truncation error size can be complex, and therefore, the nonlinearity between the two factors should be considered.

5.2.1 Implied volatility level and truncation error

Section 2.2.3 shows that if the truncation is ignored and the fair value of the contracts V, W, and X are estimated using only the OTM option prices for the strike price domain $[K_{\min}(t,\tau), K_{\max}(t,\tau)]$ for time t and maturity τ, it is equivalent to assuming that

$$\mathbb{P}_t \left\{ \ln\left[\frac{S(t+\tau)}{S(t)}\right] < \ln\left[\frac{K_{\min}(t,\tau)}{S(t)}\right] \right\} = 0 \tag{5.1}$$

and

$$\mathbb{P}_t \left\{ \ln\left[\frac{S(t+\tau)}{S(t)}\right] > \ln\left[\frac{K_{\max}(t,\tau)}{S(t)}\right] \right\} = 0, \tag{5.2}$$

where $S(t)$ is the underlying price level at time t, for any probability measure $\mathbb{P}$ for which the fair value is estimated. The assumptions in Equations (5.1) and (5.2) suggest that the implied moment estimate under truncation is related to the corresponding moment of the truncated probability density that is defined by the function

$$g(x) = \begin{cases} 0, & \text{if } x < \ln(K_{\min}(t,\tau)/S(t)) \text{ or } x > \ln(K_{\max}(t,\tau)/S(t)); \text{ and} \\ z^{-1}f(x), & \text{if } \ln(K_{\min}(t,\tau)/S(t)) \leq x \leq \ln(K_{\max}(t,\tau)/S(t)), \end{cases} \tag{5.3}$$

where $f(x)$ is the implied RND function, and

$$z = \int_{\ln(K_{\min}(t,\tau)/S(t))}^{\ln(K_{\max}(t,\tau)/S(t))} f(x)dx. \tag{5.4}$$

This relationship implies that the truncation of OTM option prices on the strike price domain of the OTM option price function can be translated into the truncation of the mass of implied risk-neutral log-return density on the log-return domain of the density function, via the log-moneyness of the minimum and maximum strike prices.

Given that a density becomes fatter and wider as its volatility increases, the difference between the probability densities defined by $f(x)$ and $g(x)$ can be more significant if the volatility of the former is higher, even when the points of truncation, i.e., $\ln(K_{\min}(t,\tau)/S(t))$ and $\ln(K_{\max}(t,\tau)/S(t))$, are fixed. If the volatility is low so that only the thin part of RND is truncated, then the difference between $f(x)$ and $g(x)$ will be insignificant. In contrast, if the volatility is high so that a big mass of RND is truncated, then $g(x)$ will be considerably different from $f(x)$. Jiang and Tian (2005) reflect this property by expressing the minimum and maximum strike prices as multiples of standard deviations while introducing truncation errors, and this approach implies that there is a need to consider the level of implied volatility when analysing the relationship between truncation level and truncation error size.

A simple test can be conducted as follows to demonstrate the effect of implied volatility level on truncation error size. First, multiple sets of OTM option prices are generated using BS model while setting the volatility parameter σ differently. Next, the trajectory of the implied moment estimates for each option price set are examined while changing the width of the integration domain in order to investigate whether the estimate converges to the true value at a uniform speed regardless of the implied volatility level. If the difference in the implied volatility level does not affect the relationship between the truncation level and the truncation error size, then the estimate will converge to true value as domain width increases at a uniform speed for any implied volatility level. In this chapter, we conduct this test using five sets of OTM model option prices, for each of which σ is set as 0.1, 0.2, 0.3, 0.4, and 0.5, respectively. The underlying price, time to maturity, and risk-free rate are set as 1183.2, 0.25, and 0.0301, respectively, as in Section 2.4.2. In this case, the implied volatility will be equal to σ, whereas implied skewness and kurtosis will stay at zero and three, respectively, regardless of how we set σ. The integration domain is set to maintain the symmetry in terms of log-moneyness, i.e., $\ln(K_{\min}(W)/S) = -\ln(K_{\max}(W)/S)$, for each domain half-width $W = 0.5(K_{\max} - K_{\min})$ given the findings in Chapter 4.

Figure 5.1 depicts the test result. The figure shows that the implied moment estimate converges to the true value more slowly as implied volatility level becomes higher. Especially, it is notable in Figures 5.1b and 5.1c that the speed of convergence is different even when the true level of the corresponding moment is fixed. The results reveal that the relationship between truncation level and truncation error size is affected by the level of implied volatility, and therefore there is a need to modify the definition of truncation level based on the level of implied volatility in order to explain the relationship appropriately.

The impact of the implied volatility level on the truncation error size is an empirically important issue given the evidence from the S&P 500 index options market that there is a strong relationship between the integration domain width and the implied volatility level. Figure 5.2 illustrates the time-series dynamics of the integration domain width in terms of strike price and the implied volatility estimate during the period from 2000 to 2010. As the figure suggests, the minimum and maximum strike prices tend to diverge from (converge to) the underlying asset price during the high (low) implied volatility period. Does this imply that more accurate estimates of implied moments can be obtained during the high implied volatility periods since truncation is less intensive in this period? The results in this subsection suggest that this is not the case. Namely, even when the minimum and maximum strike prices tend to diverge from the underlying price, the range of the implied RND covered by the OTM option prices can be narrower if the implied volatility level becomes higher. Hence, it can be inappropriate to conclude that a more complete OTM option availability does always lead to a more accurate implied moment estimation.

5.2.2 Nonlinearity in the relationship

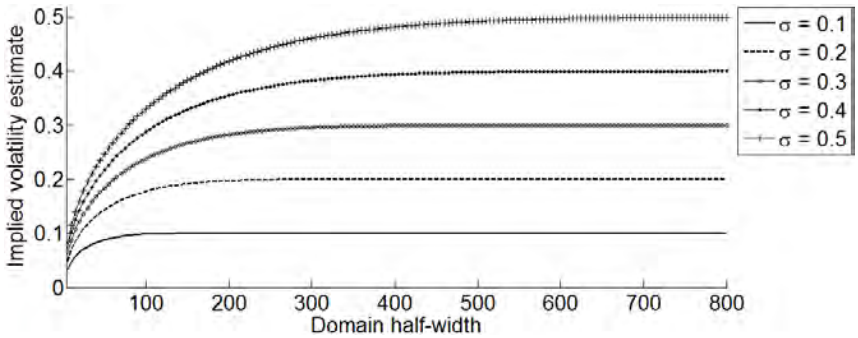
Chapter 4 shows that the size of truncation error is related to the width and asymmetry level of integration domain, and the relationship among the three factors is shown to be nonlinear. The nonlinear relationship can be summarised as two components. First, the relationship between the integration domain width and the truncation error size tends to be concave, i.e., the change in the size of truncation error after an inclusion or exclusion of option prices on the outermost part of the integration domain tends to be smaller when the integration domain is wider. This is intuitive because the included or excluded options will be deeper-OTM when the integration domain is wider, and therefore the corrsponding option prices will be cheaper. Second, the impact of the integration domain asymmetry level on the truncation error size tends to be smaller when the integration domain is wider. This is also natural because deeper-OTM option prices will be newly included and excluded by a change in the domain asymmetry level when the integration domain is wider.

Figure 5.1: Impact of implied volatility level on truncation error

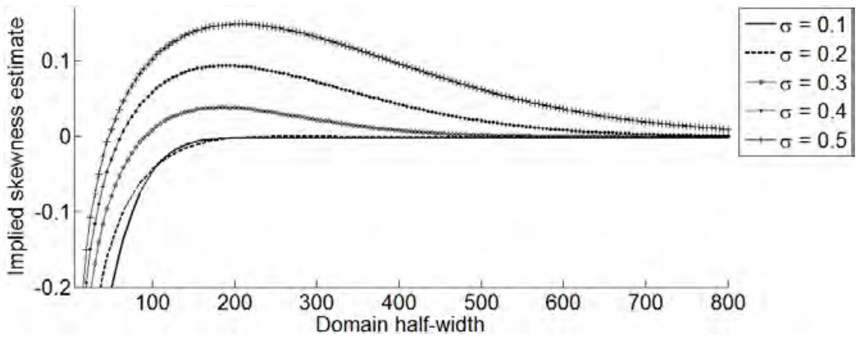
This figure illustrates the relationship between the implied volatility level and the truncation error size by showing how quickly the implied moment estimate converges to the true value as the integration domain width increases at different implied volatility levels. BS model is used to generate option prices. For each domain half-width W, the minimum and maximum strike prices are set to make the domain symmetric in terms of log-moneyness, i.e.,

$$\ln(S/K_{\min}) = \ln(K_{\max}/S); \quad \text{and} \quad K_{\max} - K_{\min} = 2W,$$

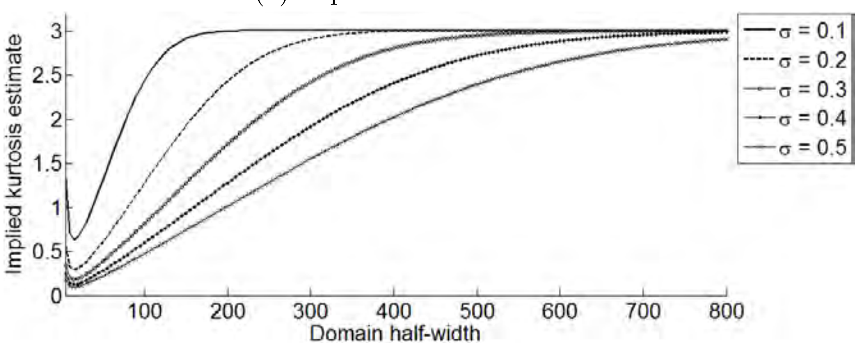
where $S = 1178.3$ is the dividend-adjusted underlying price, and $K_{\min}$ and $K_{\max}$ are the minimum and maximum strike prices, respectively. The level of implied volatility, skewness, and kurtosis are depicted in Figures 5.1a, 5.1b, and 5.1c, respectively. The strike price interval is set as 0.1.



(a) Implied volatility estimate



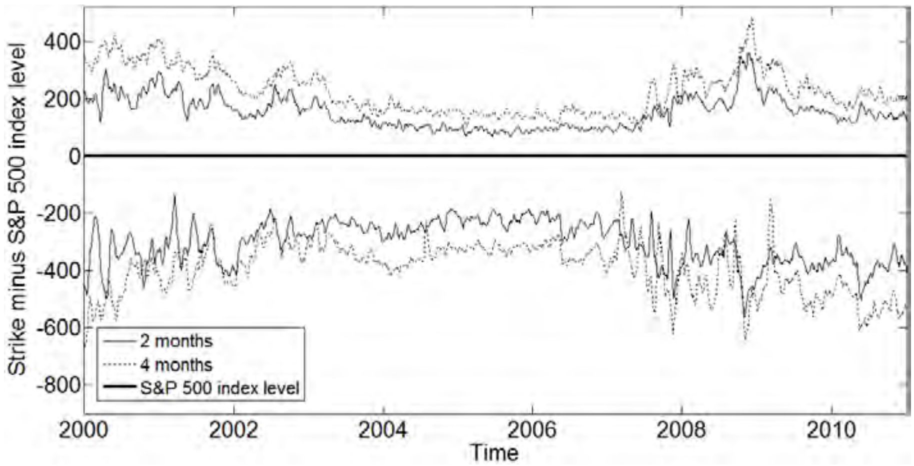
(b) Implied skewness estimate



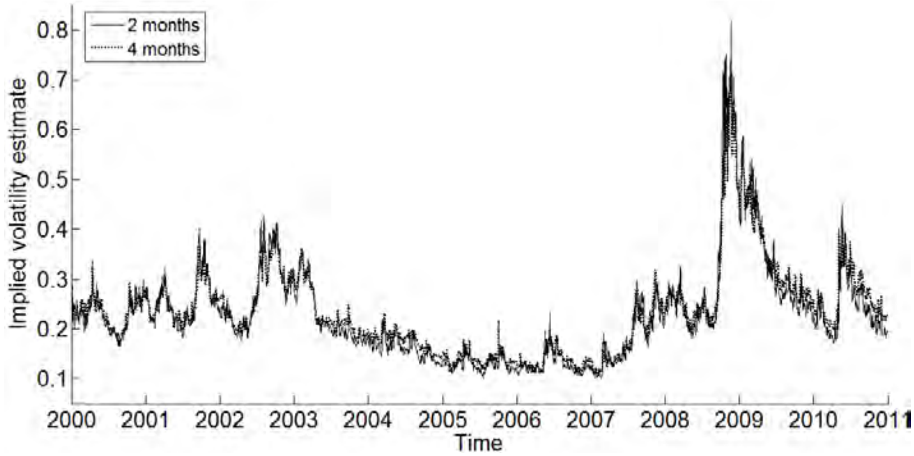
(c) Implied kurtosis estimate

Figure 5.2: Strike price domain width and implied volatility level

This figure shows the time-series dynamics of strike price domain width and implied volatility level in the S&P 500 index options dataset, which spans a time period from January 2000 to December 2010. Strike price domain and implied volatility level are measured based on implied volatility curves for time to maturity of two and four months, which are extracted from daily implied volatility surfaces. Figure 5.2a illustrates the distance of minimum and maximum strike prices from daily index level after sample filtration. Figure 5.2b depicts daily implied volatility level, which is estimated using the implied volatility estimator of Bakshi et al. (2003) and linear extrapolation method. Linear extrapolation is applied up to the points where strike prices are $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-free index level at day t.



(a) Difference from S&P 500 index level



(b) Implied volatility estimate

The two nonlinear properties above suggest that the explanatory power of the truncation level with respect to the truncation error size can be weak even when the level is quantified appropriately, if those properties are not considered while modelling the relationship between truncation level and truncation error size. Hence, this chapter introduces some additional variables for the regression analysis in Section 5.4 to consider the nonlinearity. First, the natural logarithm of integration domain width is employed to reflect the concave relationship between domain with and truncation error size. Second, the product of domain width and asymmetry level is included to consider the interaction between the two factors.

5.3. Domain stabilisation

This section introduces the new truncation treatment method, i.e., DStab, and elaborates on its characteristics based on the findings in Section 5.2. Section 5.3.1 describes the definition of DStab and explains its rationale. Section 5.3.2 addresses some issues regarding DStab and shows how they can be resolved. Section 5.3.3 demonstrates an example of implementing DStab and discusses the consequences of DStab.

5.3.1 Concept and definition

Section 5.2.1 proposes IVAL as a variable that can explain the relationship between truncation and truncation error well. If this is the case, it can be possible to control the magnitude of truncation error by controlling the level of truncation that is defined in terms of IVAL. Since the direction of causality between truncation and truncation error is clear, the size of truncation error may be controlled by manipulating the width and asymmetry level of the integration domain, which are shown by Chapter 4 to be the two variables that can characterize the level of truncation, are defined in terms of IVAL and then controlled.

The new truncation treatment method proposed in this paper, i.e., DStab, is based on the ideas of 1) measuring the minimum and maximum values of the integration domain in terms of IVAL, and 2) stabilising the two endpoint values by discarding some of the OTM option prices that are available. If the minimum and maximum IVALs are maintained at a fixed level either cross-sectionally or over time, then the width and asymmetry level of integration domain will be stabilised in terms of IVAL, and therefore the size of truncation error will become less volatile on the condition that the truncation level in terms of IVAL is closely related to the size of truncation error. Although it is impossible to fix the size of truncation error because the error is also related to the shape of RND, it is likely that the error will become less volatile by DStab because the other major factor of truncation error, i.e., the level of truncation, is stabilised.

5.3.2 Issues

There are three issues regarding DStab. First, although DStab may stabilize the size of truncation error, it is also likely that DStab will increase the size in mean. As argued in Chapter 4, a truncation treatment method that conducts a further reduction of integration domain may induce an increase in the size of truncation error, because the reduction will make the width of integration domain smaller. Hence, DStab can be regarded as a tradeoff between the mean and volatility of truncation error. This is the reason why one first needs to clarify the objective of implied moment estimation before conducting DStab. If the estimate needs to be as close as possible to the true value, regardless of how volatile the estimation error is, then DStab will not be a good truncation error reduction method to employ. On the other hand, if truncation error needs to be less volatile in order to make a comparison among implied moment levels more accurate, then DStab can be considered as a good option for truncation treatment.

Although less volatile truncation error will make it easier to discern differences in implied moments across observation, there is a possibility that moment estimate will convey misleading information if truncation error is too large. For instance, a leptokurtic density can be estimated as platykurtic if truncation error in implied kurtosis estimate is too large. In order to avoid getting affected by such large truncation errors, we suggest employing LE in conjunction with domain stabilisation method. As shown in Chapter 3, LE significantly reduces the truncation error regardless of the implied moment being estimated, although the reduction is incomplete. Hence, combining the two methods can make truncation error moderately large and less volatile.

Second, there is no definite rule for setting the threshold volatility-adjusted log-moneyness levels. It is difficult to decide at which point options should be trimmed off, since there is a tradeoff between better stabilisation and smaller truncation error, i.e., we need to discard more options if we want to stabilise the truncation level more strongly via a more intensive trimming. In order to set a criterion, we first determine the percentage of endpoint volatility-adjusted log-moneyness values that will be fixed after stabilisation, and then choose the threshold values based on the percentage level. For instance, if we want 90 percent of volatility-adjusted endpoint log-moneyness levels to be fixed, then we set the threshold levels to be the 90$^{\text{th}}$ percentile of volatility-adjusted minimum log-moneyness and 10$^{\text{th}}$ percentile of volatility-adjusted maximum log-moneyness in the entire sample.

Finally, the level of implied volatility, what is to be estimated, is required *ex ante*. This issue can be circumvented by conducting estimation in two stages. Chapter 3 shows that the impact of truncation on implied volatility estimator is much smaller than implied skewness or kurtosis estimator. Hence, we first estimate implied volatility without DStab but only with LE, and then take the estimate for volatility adjustment. Next, we conduct DStab that is followed by another set of LE, and then estimate the implied skewness and kurtosis.

5.3.3 Example of implementation

Figure 5.3 illustrates the level of volatility-adjusted endpoint log-moneynesses of S&P 500 index options with 2 months to maturity before and after domain width stabilisation. Two sets of stabilisation are done in order to fix 90 and 99 percent of volatility-adjusted endpoint log-moneyness values, respectively. When Figure 5.3b is compared to Figure 5.3c, it can be found that domain width is larger in the case of 90 percent stabilisation than 99 percent stabilisation, but volatility-adjusted endpoint log-moneyness levels are fixed more firmly in the case of 99 percent stabilisation. This again shows that one faces a tradeoff when deciding the threshold values for domain width stabilisation.

5.4. Empirical analysis

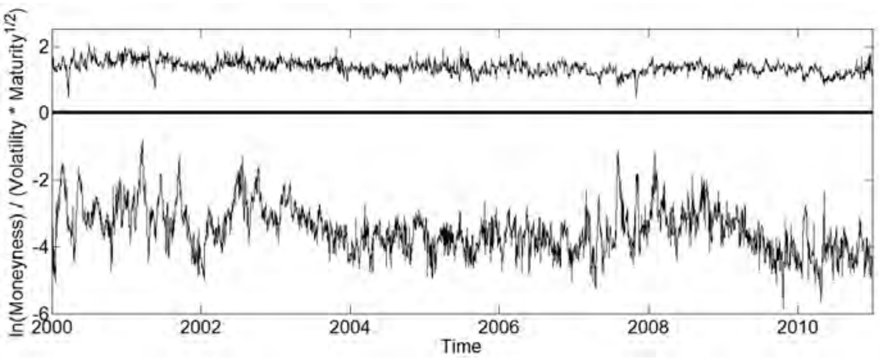
This section reports the result of empirical analysis on S&P 500 index options to show that 1) the relationship between the truncation level and the truncation error size is best explained when the truncation level is defined in terms of IVAL, and 2) DStab makes the size of truncation error less volatile. Section 5.4.1 describes how option prices are estimated for fixed maturities by generating implied volatility surfaces. Section 5.4.2 investigates whether the level of truncation has the highest explanatory power with respect to the size of truncation error when it is measured in terms of IVAL rather than strike price, moneyness, or unadjusted log-moneyness. Section 5.4.4 examines whether DStab can make the size of truncation error less volatile, and if so, the volatility reduction effect can also be found from other similar methods. A description of the S&P 500 index options data used in this chapter is provided in Section 2.3.

Figure 5.3: Impact of domain stabilisation

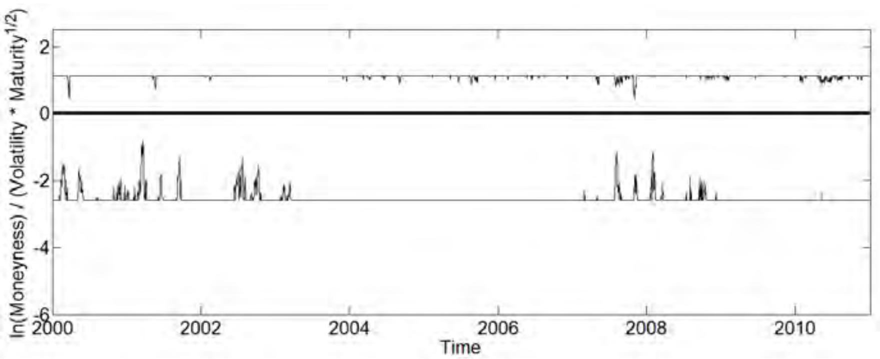
This figure illustrates the impact of domain stabilisation by showing how the minimum and maximum volatility-adjusted log-moneyness is changed after stabilisation. For minimum and maximum strike prices at each time t, volatility-adjusted log-moneyness is defined as

$$\frac{\ln(K/S(t))}{\mathrm{VOL}(t,\tau)\sqrt{\tau}},$$

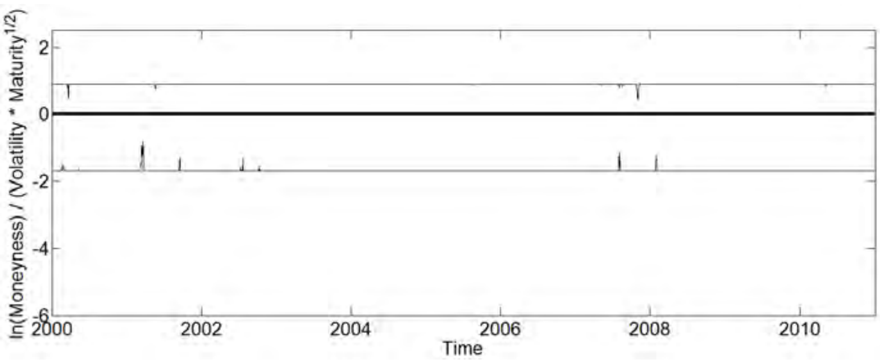
where K is the strike price, $S(t)$ is the dividend-free index level at time t, $\mathrm{VOL}(t,\tau)$ is implied volatility level at time t and time to maturity τ. S&P 500 index options dataset, which spans a time period from January 2000 to December 2010, is used to generate daily implied volatility surfaces, from which minimum and maximum volatility-adjusted log-moneyness values are extracted for $\tau = 2$ months. An n-percent stabilisation is done by discarding options whose volatility-adjusted log-moneyness is smaller than the n^{th} percentile of minimum volatility-adjusted log-moneyness values in the sample, or larger than the $(100-n)^{\text{th}}$ percentile of maximum volatility-adjusted log-moneyness values. Implied volatility level is estimated using the implied volatility estimator of Bakshi et al. (2003) and linear extrapolation method. Linear extrapolation is applied up to the points where strike prices are $S(t)/3$ and $3S(t)$, respectively.



(a) Before stabilisation



(b) After 90 percent stabilisation



(c) After 99 percent stabilisation

5.4.1 Generation of implied volatility surface

For empirical analysis, we generate daily Black-Scholes implied volatility surfaces as in Chapter 3 to avoid the telescoping problem. In order to generate a surface, we first construct Black-Scholes implied volatility curves for maturities with available option prices. With these volatility curves, we generate an implied volatility surface by estimating a bicubic spline function. If option prices are originally unavailable for a maturity, the minimum and maximum strike prices are approximated by linearly interpolating the corresponding endpoint strike price for the two nearest maturities for which option prices are available.

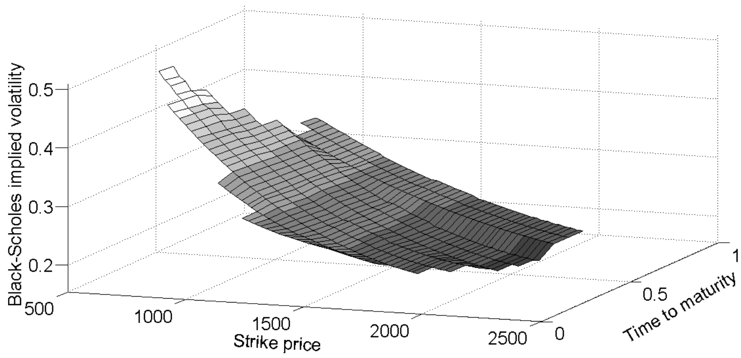
When the implied moments are estimated, we first extract an implied volatility curve from the daily surface by generating implied volatility observations between the minimum and maximum strike prices with a strike price gap of 0.1 using the bicubic spline function, and then convert the implied volatility observations to OTM option prices. When LE is employed, we extrapolate the two endpoint implied volatility levels of this extracted curve up to the points where the strike prices are $S(t)/3$ and $3S(t)$, respectively, where $S(t)$ is the dividend-adjusted index level on day t. The strike price interval between the option prices generated by LE is also set as 0.1. Figure 5.4 shows an example of daily implied volatility surface and the corresponding OTM option prices.

5.4.2 Nonlinear relationship between truncation level and truncation error size

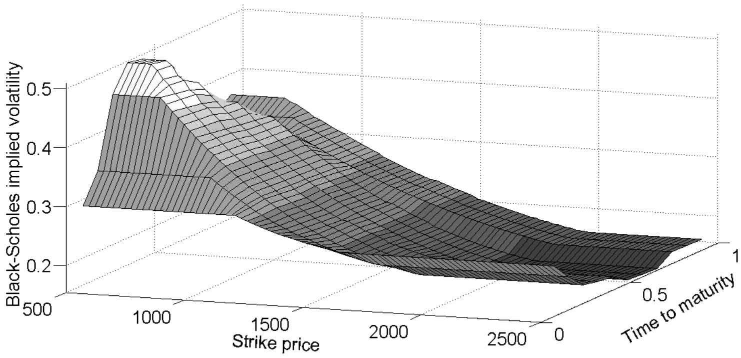
Before investigating whether the level of truncation is more closely related to the size of truncation error when the level is measured in terms of IVAL, we first examine if the relationship between the truncation and the truncation error is in fact nonlinear as suggested in Section 5.2.2. If this is the case, the nonlinearity should be considered to make the comparison among the different units for measuring truncation level more complete. To consider the nonlinearity, we include the natural logarithm of domain width and the product of domain width and asymmetry level and check if these new variables can increase the explanation power of the truncation level with respect to the truncation error size.

Figure 5.4: Example of implied volatility and option price surfaces

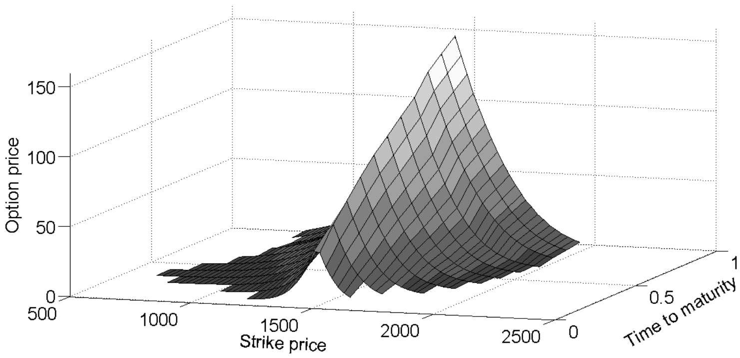
This figure shows an example of implied volatility surface and the corresponding option prices, which is constructed based on the out-of-the-money option prices on January 3rd, 2000. Strike price gap is set to be twenty-five for a better visualization. Implied volatility surfaces before and after linear extrapolation are depicted in Figure 5.4a and 5.4b, respectively. Out-of-the-money option price levels corresponding to Figures 5.4a and 5.4b are illustrated in Figures 5.4c and 5.4d, respectively.



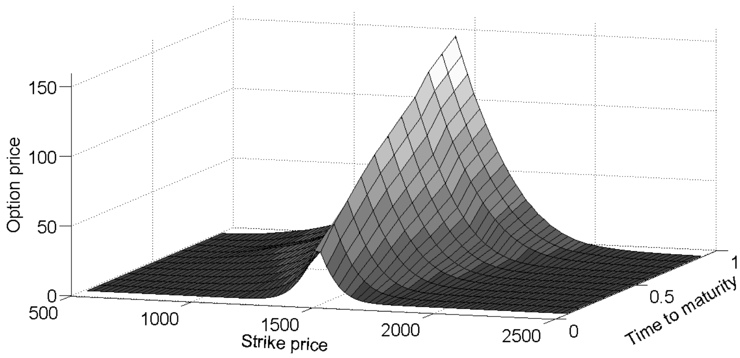
(a) Implied volatility surface (not extrapolated)



(b) Implied volatility surface (extrapolated)



(c) Option price surface derived from (a)



(d) Option price surface derived from (b)

Following the approach in Chapter 4, the width and asymmetry level of integration domain are defined as

$$\text{Width}(t,\tau) = X_{\max}(t,\tau) - X_{\min}(t,\tau), \quad \text{Asym}(t,\tau) = \ln\left(\frac{X_{\max}(t,\tau) - X_{\text{ATM}}(t)}{X_{\text{ATM}}(t) - X_{\min}(t,\tau)}\right), \tag{5.5}$$

where $X_{\min}(t,\tau)$, $X_{\max}(t,\tau)$, and $X_{\text{ATM}}(t)$ are the minimum value, the maximum value, and the at-the-money value of the integration domain for time t and maturity τ, respectively, that are measured in terms of the corresponding unit. For instance, if the truncation level is measured in terms of strike price, the definitions in Equation (5.5) become

$$\text{Width}_K(t,\tau) = K_{\max}(t,\tau) - K_{\min}(t,\tau), \quad \text{Asym}_K(t,\tau) = \ln\left(\frac{K_{\max}(t,\tau) - S(t)}{S(t) - K_{\min}(t,\tau)}\right).$$

Since the true value of the implied moments are unknown for the option prices that are observed from markets, a proxy variable needs to be employed to approximate the size of the truncation error. As in Section 4.4.2, APC is chosen as a proxy for the size of the truncation error.

Table 5.1 summarises the regression result for which the truncation level is measured in terms of IVAL.[1] Column [1] reports the benchmark result, where none of the concave relationship between the domain width and the size of truncation error or the interaction between the domain width and the domain asymmetry level is considered. Column [2] and [3] show the results where either the concavity or the interaction is considered, respectively. Finally, Column [4] reports the results where the both factors are reflected. In Table 5.1, three notable points can be found. First, Both the t-statistics for domain width and the adjusted R^2 become more significant when domain width is replaced with its natural logarithm. This implies that the relationship between domain width and turncation error size is indeed concave. Second, the coefficient estimate for the interaction term is shown to be always negative and significant, regardless of whether the concave relationship between domain width and truncation error size is considered. The negative coefficient of the interaction term can be interpreted in two ways:

- Marginal change in truncation error with respect to domain asymmetry increases

[1] The result is qualitatively equivalent when the truncation level is measured in terms of other units, and therefore only the result for IVAL is reported for brevity.

Table 5.1: Nonlinear relationship between truncation and truncation error

This table reports the regression results of truncation error proxy variable, i.e., absolute percentage change in monent estimate after linear extrapolation, on domain width and asymmetry level while considering nonlinearity in the relationship. Natural logarithm of domain width is included as an independent variable in order to consider the concave relationship between domain width and truncation error, while the interaction terms are added to take the interdependence between impact of domain width and asymmetry level on truncation error into account. Volatility-adjusted log-moneyness width and log-moneyness width log-ratio are used to measure domain width and asymmetry level, respectively. ** and * denote statistical significance at the 1% and 5% levels, respectively.

Panel A. Implied skewness estimate								
	2 months				4 months			
	[1]	[2]	[3]	[4]	[1]	[2]	[3]	[4]
Width	-0.0420^{**}		-0.2834^{**}		-0.0772^{**}		-0.4930^{**}	
	(−10.70)		(−43.44)		(−8.64)		(−40.42)	
ln(Width)		-0.2611^{**}		-1.1979^{**}		-0.4527^{**}		-1.8115^{**}
		(−14.81)		(−48.65)		(−12.75)		(−44.04)
Asymmetry	0.1819^{**}	0.1571^{**}	1.2992^{**}	1.7858^{**}	0.3600^{**}	0.3137^{**}	2.4134^{**}	2.9458^{**}
	(19.56)	(17.13)	(47.03)	(48.74)	(18.15)	(16.25)	(46.67)	(47.83)
Width×Asymmetry			-0.2548^{**}				-0.4986^{**}	
			(−41.92)				(−41.63)	
ln(Width)×Asymmetry				-1.0972^{**}				-1.8661^{**}
				(−45.27)				(−44.02)
Constant	0.4721^{**}	0.6549^{**}	1.4985^{**}	2.0057^{**}	0.8049^{**}	1.0932^{**}	2.4190^{**}	2.9017^{**}
	(31.76)	(28.60)	(55.30)	(58.13)	(24.28)	(24.22)	(51.84)	(54.05)
Adj. R^2	0.3395	0.3629	0.5971	0.6350	0.2790	0.3007	0.5578	0.5899
N	2,750	2,750	2,750	2,750	2,750	2,750	2,750	2,750

Table 5.1: Nonlinear relationship between truncation and truncation error (*cont.*)

Panel B. Implied kurtosis estimate								
	2 months				4 months			
	[1]	[2]	[3]	[4]	[1]	[2]	[3]	[4]
Width	-0.0833^{**}		-0.2190^{**}		-0.1024^{**}		-0.2469^{**}	
	(-26.53)		(-36.47)		(-31.82)		(-55.07)	
ln(Width)		-0.4307^{**}		-0.8920^{**}		-0.4813^{**}		-0.8844^{**}
		(-31.11)		(-38.28)		(-39.79)		(-58.33)
Asymmetry	0.0874^{**}	0.0688^{**}	0.7156^{**}	0.8709^{**}	0.1438^{**}	0.1241^{**}	0.8576^{**}	0.9049^{**}
	(11.74)	(9.55)	(28.14)	(25.12)	(20.14)	(18.87)	(45.11)	(39.87)
Width×Asymmetry			-0.1432^{**}				-0.1733^{**}	
			(-25.60)				(-39.36)	
ln(Width)×Asymmetry				-0.5404^{**}				-0.5536^{**}
				(-23.56)				(-35.44)
Constant	0.6990^{**}	0.9531^{**}	1.2762^{**}	1.6183^{**}	0.8206^{**}	1.0589^{**}	1.3816^{**}	1.5954^{**}
	(58.77)	(53.01)	(51.16)	(49.56)	(68.76)	(68.87)	(80.54)	(80.64)
Adj. R^2	0.4544	0.4932	0.5594	0.5782	0.5893	0.6434	0.7374	0.7552
N	2,750	2,750	2,750	2,750	2,750	2,750	2,750	2,750

(decreases) as domain width decreases (increases): This is consistent with the argument in Section 5.2.2, i.e., deeper-OTM option prices will be included or excluded by a change in the level of domain asymmetry if the integration domain is wider, so that the impact of the inclusion or exclusion will be smaller.

- Marginal change in truncation error with respect to domain width increases (decreases) as domain becomes more biased to the OTM call (put) side: This is consistent with Chapter 4 which shows that the magnitude of change in implied skewness estimate with after a change in domain width tends to be larger (smaller) when the integration domain is more biased to the OTM call (put) side, when the true implied skewness is negative. Several studies report that the implied skewness is significantly negative in the S&P 500 index options market.

Finally, when both of the two new variables are included as in Column [4], explanatory power of the model is considerably increased when compared to Column [1]. The level of adjusted R^2 is ranged between 0.578 and 0.755 in Column [4], whereas its value is between 0.279 and 0.589 in Column [1]. The increase in the level of adjusted R^2 suggests that a large part of truncation error size can be explained with the level of truncation if measured properly and the nonlinear relationship between truncation level and truncation error size is considered.

5.4.3 Measuring unit of truncation level

Section 5.4.2 shows how the relationship between truncation level and truncation error size should be modeled while considering the nonlinearity. Based on this finding, this subsection investigates how the level of truncation should be defined to maximise its explanatory power on the size of truncation error. The following two questions should be answered to define the level of truncation:

- How should the minimum and maximum points of the integration domain be measured and quantified? Namely, among the strike price, moneyness, and log-moneyness of the end-points, which one should be considered?

- Should the level of implied volatility considered when measuring the level of truncation?

To answer the two questions above, we compare six cases where the strike price, moneyness, and log-moneyness are used with and without volatility adjustment to define the width and asymmetry level of the integration domain, respectively. Specifically, we investigate in which case the two truncation level variables have the strongest explanatory power on the size of truncation error, using the regression model with the two nonlinearity-reflected variables in Section 5.4.2.

Table 5.2 reports the regression result. The tables reveals two interesting findings. First, the explanatory power of the width and asymmetry level of the integration domain increases significantly after volatility adjustment. This suggests that the level of implied volatility should be considered to explain the relationship between the truncation level and the truncation error size. Second, when implied volatility level is considered, the explanatory power is the strongest when log-moneyness is used to measure the width and asymmetry level of the integration domain, regardless of the maturity and moment considered. This is consistent with Section 2.2.3 which shows that the truncation of integration is linked to the truncation of the implied risk-neutral log-return density via the end-point log-moneyness of the integration domain. Overall, the results in Table 5.2 suggest that the minimum and maximum points of the integration domain should be measured in terms of IVAL to explain relationship between truncation level and truncation error size.

5.4.4 Effectiveness of DStab

In this subsection, we empirically test whether DStab can reduce volatility of truncation error effectively, and compare the effectiveness with some alternative methods. In addition to comparing preliminary statistics before and after truncation treatment, we also conduct a set of variance comparison tests, using the test statistic of Levene (1960) which is robust to nonnormailty, and the two alternative statistics of Brown and Forsythe (1974), which are shown to be even more robust when dealing with skewed distributions. If a method is effective in reducing volatility of truncation error, the test statistics will indicate a significant decrease in variance after DStab.

The alternative methods, to which DStab is compared, are chosen in order to achieve two goals. First, two of the alternatives are chosen so that by comparing them to DStab one can find out whether the main features of DStab (use of log-moneyness-based measure and volatility adjustment) in fact enhance effectiveness. Specifically, we choose two different versions of DSym, for one of which log-moneyness is replaced with moneyness K/S, while volatility adjustment is omitted for the other. Second, the other two alternatives are chosen to compare DStab to DSym. Two different types of symmetrisation methods, for each of which strike price and log-moneyness are used to define symmetry, respectively, are employed.

Table 5.3 reports the test result. Three interesting points can be found here. First, truncation error becomes less volatile after DStab. In Panel A, standard deviation is found to decrease after DStab. Furthermore, in Panel B, variance comparison test result confirm that the decrease in standard deviation is statistically significant, both for the implied skewness and kurtosis estimators. Second, DStab becomes less effective or even ineffective if either log-moneyness-based measure or volatility adjustment is not applied. In Panel B, it is shown that the decrease in the truncation error volatility becomes statistically insignificant if log-moneyness is replaced with moneyness, when 90 percent DStab is conducted for $\tau = 2$ months, regardless of the moment being estimated. On the other hand, the truncation error volatility of the implied kurtosis estimate is found to increase after DStab if volatility adjustment is not applied. Finally, both the mean and volatility of the truncation error increase after DSym regardless of the moment being estimated and the definition of symmetry being applied. A possibile reason for this is that while DSym fixes the asymmetry level of the integration domain, it does not control the width. As mentioned in Sections 4.3 and 4.4, the width of the integration domain is closely related to the truncation error size even for the implied skewness estimator. Furthermore, the truncation error is found to be smaller after DStab than after DSym. In Panel A, it is shown that mean of truncation error is smaller for either 90 or 99 percent DStab, when compared to DSym. This is because DStab method minimises the number of discarded options by relaxing the domain symmetry requirement.

Table 5.2: Explanatory power of truncation level with respect to truncation error size

This table reports the regression results of the proxy variable for the size of truncation error, i.e., the absolute percentage change in monent estimate after linear extrapolation, on truncation level while considering the nonlinearity in the relationship. The width and asymmetry level of integration domain are measured using different units, i.e., strike price, moneyness, and log-moneyness with and without implied volatility adjustment. The abbreviations IVAS, IVAM, IVAL stand for implied volatility adjusted strike price, implied volatility adjusted moneyness, and implied volatility adjusted log-moneyness, respectively. For each unit, domain width and asymmetry level are defined as

$$\text{Width}(t,\tau) = X_{\max}(t,\tau) - X_{\min}(t,\tau), \quad \text{Asym}(t,\tau) = \ln\left(\frac{X_{\max}(t,\tau) - X_{\text{ATM}}(t)}{X_{\text{ATM}}(t) - X_{\min}(t,\tau)}\right),$$

where $X_{\min}(t,\tau)$, $X_{\max}(t,\tau)$, and $X_{\text{ATM}}(t)$ are the minimum value, the maximum value, and the at-the-money value of the integration domain for time t and maturity τ, respectively, that is measured in terms of the corresponding unit. ** and * denote statistical significance at the 1% and 5% levels, respectively.

	Strike Price	Moneyness	Log-moneyness	IVAS	IVAM	IVAL
Panel A. Implied skewness estimate (maturity of two months)						
ln(Width)	-0.3453^{**}	-0.1514^{**}	-0.2615^{**}	-0.4234^{**}	-1.0706^{**}	-1.1979^{**}
	(-15.11)	(-8.88)	(-14.00)	(-24.44)	(-42.27)	(-48.65)
Asym	-0.0263	0.1487^{**}	0.0271	1.1735^{**}	1.9098^{**}	1.7858^{**}
	(-1.09)	(6.69)	(1.29)	(30.02)	(42.60)	(48.74)
ln(Width)×Asym	-0.3507^{**}	-0.1153^{**}	-0.2560^{**}	-0.6085^{**}	-1.2343^{**}	-1.0972^{**}
	(-12.01)	(-5.16)	(-11.24)	(-25.16)	(-39.38)	(-45.27)
Constant	0.0102	0.1420^{**}	0.0978^{**}	0.8961^{**}	1.7135^{**}	2.0057^{**}
	(0.57)	(9.07)	(7.62)	(34.60)	(49.84)	(58.13)
Adj. R^2	0.3523	0.3228	0.3386	0.4282	0.5752	0.6350
N	2,750	2,750	2,750	2,750	2,750	2,750
Panel B. Implied skewness estimate (maturity of four months)						
ln(Width)	-1.0129^{**}	-0.2821^{**}	-0.4986^{**}	-0.7929^{**}	-1.5827^{**}	-1.8115^{**}
	(-29.02)	(-8.35)	(-11.62)	(-33.58)	(-38.20)	(-44.04)
Asym	-0.2013^{**}	0.4029^{**}	0.1829^{**}	2.3729^{**}	3.9077^{**}	2.9458^{**}
	(-6.18)	(11.43)	(6.44)	(42.43)	(41.37)	(47.83)
ln(Width)×Asym	-1.2435^{**}	-0.1478^{**}	-0.5022^{**}	-1.3689^{**}	-2.1456^{**}	-1.8661^{**}
	(-23.46)	(-3.09)	(-11.89)	(-37.45)	(-38.14)	(-44.02)
Constant	-0.1096^{**}	0.2413^{**}	0.2559^{**}	1.4437^{**}	2.9323^{**}	2.9017^{**}
	(-5.62)	(11.18)	(8.93)	(44.18)	(43.59)	(54.05)
Adj. R^2	0.4168	0.2681	0.2960	0.5194	0.5294	0.5899
N	2,750	2,750	2,750	2,750	2,750	2,750
Panel C. Implied kurtosis estimate (maturity of two months)						
ln(Width)	-0.2364^{**}	-0.1319^{**}	-0.2098^{**}	-0.2678^{**}	-0.8671^{**}	-0.8920^{**}
	(-11.88)	(-8.95)	(-10.17)	(-16.22)	(-36.63)	(-38.28)
Asym	0.1221^{**}	0.2030^{**}	0.0726^{**}	0.5821^{**}	1.0554^{**}	0.8709^{**}
	(5.82)	(10.57)	(3.83)	(15.63)	(25.19)	(25.12)
ln(Width)×Asym	-0.1319^{**}	-0.0335	-0.1691^{**}	-0.2715^{**}	-0.6869^{**}	-0.5404^{**}
	(-5.19)	(-1.73)	(-8.23)	(-11.78)	(-23.45)	(-23.56)
Constant	0.1869^{**}	0.2543^{**}	-0.2341^{**}	0.7399^{**}	1.5092^{**}	1.6183^{**}
	(12.00)	(18.80)	(12.34)	(29.98)	(46.97)	(49.56)
Adj. R^2	0.3661	0.3474	0.3498	0.3299	0.5212	0.5782
N	2,750	2,750	2,750	2,750	2,750	2,750

Table 5.2: Explanatory power of truncation level with respect to truncation error size (*cont.*)

	Strike Price	Moneyness	Log-moneyness	IVAS	IVAM	IVAL
Panel D. Implied kurtosis estimate (maturity of four months)						
ln(Width)	-0.4956^{**}	-0.2679^{**}	-0.3181^{**}	-0.3431^{**}	-0.8410^{**}	-0.8844^{**}
	(−35.40)	(−19.69)	(−18.44)	(−31.12)	(−52.76)	(−58.33)
Asym	0.0434^{**}	0.1951^{**}	0.1279^{**}	0.6463^{**}	1.3160^{**}	0.9049^{**}
	(3.33)	(13.75)	(11.21)	(24.74)	(36.22)	(39.87)
ln(Width)×Asym	-0.4618^{**}	-0.1792^{**}	-0.2815^{**}	-0.3118^{**}	-0.6956^{**}	-0.5536^{**}
	(−21.73)	(−9.30)	(−16.57)	(−18.26)	(−32.15)	(−35.44)
Constant	0.1392^{**}	0.2320^{**}	0.2910^{**}	0.8352^{**}	1.7327^{**}	1.5954^{**}
	(17.78)	(26.67)	(25.25)	(54.73)	(66.96)	(80.64)
Adj. R^2	0.5877	0.4780	0.5000	0.5393	0.6940	0.7552
N	2,750	2,750	2,750	2,750	2,750	2,750

Table 5.3: Impact of domain width stabilisation on truncation error

This table shows how the mean and variance of truncation error are changed after applying a number of different truncation treatment methods, by reporting the sample mean and variance of proxy variable for truncation error, i.e., absolute percentage change in monent estimate after linear extrapolation, before and after treatment, as well as a set of variance comparison test statistics. An n-percent stabilisation is done by discarding options whose location on integration domain (in terms of the measures used in this table) is more left-sided than the n^{th} percentile of left-side domain endpoints in the sample, or more right-sided than the $(100-n)^{\text{th}}$ percentile of right-side domain endpoints. On the other hand, domain symmetrisation is done by discarding some options so that the size of intergration domain is maximised while satisfying one of the following conditions:

$$\begin{cases} S - K_{\min} = K_{\max} - S, & \text{(Strike-price-based symmetrisation)} \\ \ln(S/K_{\min}) = \ln(K_{\max}/S), & \text{(Log-moneyness-based symmetrisation)} \end{cases}$$

where S is the underlying price, $K_{\min}$ is the minimum strike price after symmetrisation, and $K_{\max}$ is the maximum strike price after symmetrisation. In order to minimise the impact of outliers, a daily observation is discarded if the value of proxy variable is larger than 1,000 percent for any method or measure. Mean and standard deviation of truncation error proxy variable are reported in Panel A. In Panels B and C, three different types of variance comparison test statistics are presented to show whether the change in truncation error variance after treatmant is statistically significant, for the maturities of two and four months, respectively. In Panels B and C, a (+) mark is placed together with test statistic when variance is increased after treatment, and a (−) mark is placed when variance is decreased. ** and * denote statistical significance at the 1% and 5% levels, respectively.

Panel A. Mean and standard deviation of truncation error

Moment	Method	Measure	τ = 2 months		τ = 4 months		N
			Mean	Std. dev.	Mean	Std. dev.	
Skewness	No method applied		0.0988	0.1295	0.1272	0.2786	2,745
	90 percent stabilisation	Volatility-adjusted log-moneyness	0.1351	0.0625	0.1932	0.0688	2,745
		Volatility-adjusted moneyness	0.1562	0.0677	0.2154	0.0766	2,745
		Log-moneyness	0.1019	0.0745	0.1205	0.0829	2,745
	99 percent stabilisation	Volatility-adjusted log-moneyness	0.2633	0.0929	0.4544	0.2128	2,745
		Volatility-adjusted moneyness	0.2838	0.0900	0.4287	0.1793	2,745
		Log-moneyness	0.1191	0.0819	0.2968	0.2019	2,745
	Domain symmetrisation	Strike price	0.5658	0.1931	0.5212	0.2803	2,745
		Log-moneyness	0.7925	0.2533	0.8383	0.4026	2,745
Kurtosis	No method applied		0.2130	0.1140	0.2204	0.1515	2,745
	90 percent stabilisation	Volatility-adjusted log-moneyness	0.3473	0.0906	0.4274	0.0864	2,745
		Volatility-adjusted moneyness	0.3797	0.1015	0.4557	0.1076	2,745
		Log-moneyness	0.5094	0.3493	0.4825	0.3225	2,745
	99 percent stabilisation	Volatility-adjusted log-moneyness	0.6408	0.0661	1.0458	0.0898	2,745
		Volatility-adjusted moneyness	0.6671	0.0733	1.0348	0.0820	2,745
		Log-moneyness	0.7033	0.4627	1.1414	0.6842	2,745
	Domain symmetrisation	Strike price	0.6926	0.2232	0.6879	0.2873	2,745
		Log-moneyness	0.7944	0.2256	0.8478	0.2815	2,745

Table 5.3: Impact of domain width stabilisation on truncation error (*cont.*)

Moment	Method	Measure	Test statistics: Levene (1960) Mean-centered	Brown and Forsythe (1974) Median-centered	Trimmed-mean-centered
Panel B. Variance comparison test statistics (maturity of two months)					
Skewness	90 percent stabilisation	Volatility-adjusted log-moneyness	(−) 8.90**	(−) 1.65	(−) 1.81
		Volatility-adjusted moneyness	(−) 1.21	(−) 0.17	(−) 0.18
		Log-moneyness	(−) 0.17	(−) 2.88	(−) 2.80
	99 percent stabilisation	Volatility-adjusted log-moneyness	(−) 56.78**	(−) 77.91**	(−) 77.89**
		Volatility-adjusted moneyness	(−) 35.67**	(−) 52.19**	(−) 52.39**
		Log-moneyness	(−) 5.09*	(−) 11.56**	(−) 11.56**
	Domain symmetrisation	Strike price	(+) 726.23**	(+) 736.61**	(+) 744.89**
		Log-moneyness	(+) 1183.09**	(+) 1106.07**	(+) 1141.15**
Kurtosis	90 percent stabilisation	Volatility-adjusted log-moneyness	(−) 35.41**	(−) 25.63**	(−) 26.78**
		Volatility-adjusted moneyness	(−) 1.73	(−) 0.54	(−) 0.57
		Log-moneyness	(+) 776.63**	(+) 567.81**	(+) 609.75**
	99 percent stabilisation	Volatility-adjusted log-moneyness	(−) 269.97**	(−) 227.20**	(−) 236.04**
		Volatility-adjusted moneyness	(−) 164.86**	(−) 135.12**	(−) 140.68**
		Log-moneyness	(+) 1007.79**	(+) 695.38**	(+) 754.20**
	Domain symmetrisation	Strike price	(+) 672.32**	(+) 652.02**	(+) 659.02**
		Log-moneyness	(+) 645.48**	(+) 606.74**	(+) 621.49**
Panel C. Variance comparison test statistics (maturity of four months)					
Skewness	90 percent stabilisation	Volatility-adjusted log-moneyness	(−) 62.64**	(−) 24.89**	(−) 27.14**
		Volatility-adjusted moneyness	(−) 44.26**	(−) 15.28**	(−) 16.45**
		Log-moneyness	(−) 33.25**	(−) 8.91**	(−) 10.08**
	99 percent stabilisation	Volatility-adjusted log-moneyness	(−) 247.22**	(−) 317.29**	(−) 315.94**
		Volatility-adjusted moneyness	(−) 100.09**	(−) 149.86**	(−) 148.30**
		Log-moneyness	(−) 112.32**	(−) 129.28**	(−) 138.09**
	Domain symmetrisation	Strike price	(+) 156.12**	(+) 190.94**	(+) 191.83**
		Log-moneyness	(+) 577.59**	(+) 518.23**	(+) 564.96**
Kurtosis	90 percent stabilisation	Volatility-adjusted log-moneyness	(−) 175.06**	(−) 111.35**	(−) 122.32**
		Volatility-adjusted moneyness	(−) 46.55**	(−) 25.66**	(−) 27.96**
		Log-moneyness	(+) 622.74**	(+) 494.56**	(+) 529.30**
	99 percent stabilisation	Volatility-adjusted log-moneyness	(−) 121.50**	(−) 75.78**	(−) 87.53**
		Volatility-adjusted moneyness	(−) 219.86**	(−) 142.31**	(−) 156.24**
		Log-moneyness	(+) 1497.43**	(+) 1107.07**	(+) 1190.66**
	Domain symmetrisation	Strike price	(+) 530.97**	(+) 535.44**	(+) 539.51**
		Log-moneyness	(+) 518.05**	(+) 504.26**	(+) 512.16**

Overall, the results in Table 5.3 show that DStab reduces volatility of truncation error effectively while minimising the size of additional truncation error that is caused by discarding observations. Especially, DStab is shown to be significantly effective even when it is used for implied kurtosis estimation. Given that no truncation error treatment method has been suggested for implied kurtosis estimation so far, reliability of implied kurtosis estimator can therefore be increased by this method. Furthermore, DStab is also shown to be more effective for controlling truncation error of implied skewness estimate, when compared to DSym, which makes DStab an even more attractive option for truncation error treatment.

5.5. Conclusion

Option price quotes are available only for a limited number of strike prices in most markets, and sometimes this is different from what is assumed in a model. With this limitation, for instance, it is impossible to obtain option prices for a continuum of strike prices that span the entire positive real line, which are assumed available in Bakshi et al. (2003). As pointed out by Jiang and Tian (2005), the limited availability of option prices induce two implementation issues, i.e., strike price discreteness and truncation. Although it is relatively easier to mitigate strike price discreteness using techniques such as interpolation, it is more difficult to deal with truncation because information is more limited for the option prices beyond the strike price domain that is covered by available option prices.

Given that ignoring the truncation and integrating the weighted option prices on a domain of finite length will make the implied moment estimators of Bakshi et al. (2003) biased, a proper treatment is needed to alleviate the estimation bias, which is called 'truncation errors' by Jiang and Tian (2005). Chapters 2 and 3 show that the implied skewness and kurtosis estimators, which rely more heavily on DOTM option prices, are more exposed to such errors, and therefore it can be conjectured that truncation should be treated more carefully for those higher moment estimators than implied volatility estimator. Although there exist two truncation error reduction methods, i.e., LE and DSym, their effectiveness is reported by Chapters 3 and 4 to be conditional or limited, especially for the implied skewness and kurtosis estimators. Hence, there is still a need for an alternative methodology with which the impact of truncation on the implied higher moment estimators can be reduced more effectively.

This chapter suggests a new truncation treatment method, i.e., DStab, which reduces the volatility of truncation errors. This approach is totally different from LE and DSym which focus on reducing the size of truncation errors. The rationale behind DStab is that the de facto impact of truncation on implied moment estimation is diminished if the size of truncation error is maintained at a fixed level cross-sectionally and over time, regardless how high the fixed level is, when the estimation is conducted to track the dynamics of implied moment over time or make a cross-sectional comparison of implied moment level across options on different underlying assets. By taking this approach, impact of truncation on implied moment estimation can be reduced without executing the extremely difficult task of estimating the OTM option prices that are not observable.

DStab reduces the volatility of truncation error by first characterizing the relationship between the level of truncation and the size of truncation error, and then manipulating the former accordingly to stabilize the size of the latter. This study shows that the size of truncation error is closely related to the log-moneyness of the minimum and maximum strike prices that is adjusted by the level of implied volatility, and therefore DStab can make truncation error less volatile by fixing IVAL of the endpoints of the integration domain. The results of variance comparison test on the proxy variables for truncation error size suggests that the volatility of truncation error decreases after DStab while the mean increases. In contrast, the truncation error volatility reduction effect is found to be inconsistent if the specification of DStab is changed or DSym is employed instead, which implies that the volatility reduction effect of DStab is unique and, therefore, meaningful.

With DStab, this chapter does not only suggest a new way to reduce the impact of truncation on model-free implied moment estimation, but also provides a deeper understanding of the relationship between the level of truncation and the size of truncation error. In addition to the finding in Chapter 4 that truncation needs to be interpreted in terms of log-moneyness, this chapter shows that the level of implied volatility should also be considered to measure the level of truncation. By specifying the level of truncation and introducing a way to control the level, this chapter provides a methodological foundation for further studies on the higher moments of implied RND, and makes the implied skewness and kurtosis estimators of Bakshi et al. (2003) more reliable to employ.

Chapter 6

Conclusions

Options markets represent a rich source of information. We can obtain more detailed and multifaceted information about the price dynamics of an asset from its options market than from the market for the asset itself, since the option price quotes are observable for multiple strike prices and maturities. Evidence of this abundance of information can be found in the methodologies that construct the implied RND from the option prices. Given that the RND is literally the probability density of the underlying asset price at maturity under the risk-neutral measure, we can extract information about the market participants' expectations for asset price dynamics from that density. Following the seminal work of Breeden and Litzenberger (1978) who develop a simple method to synthesise the Arrow-Debreu security using option prices, research on the implied RND has continued to develop methods to collect more accurate and comprehensive information from option prices.

Unfortunately, in most options markets the availability of information is insufficient to satisfy the assumption made by the RND construction methodologies. The complete RND function can only be derived when a continuum of option prices is available for the strike prices from zero to the positive infinity, and so the methodologies presume that the continuum of option prices is readily available. However, option quotes can only be observed for a limited number of strike prices and maturities in almost every options market, and some quotes that are observed need to be discarded due to the market microstructure issues. Hence, option prices have to be approximated or, in extreme cases, ignored for the strike prices for which option prices are not observed, and this induces an estimation bias.

Although recent studies have developed methods that directly calculate the moments of the implied RND without constructing the density itself, this does not necessarily mean that we can circumvent the problem of limited option price availability using these alternative methods. For instance, the model-free implied moment estimators of Bakshi et al. (2003), which are the main topic of this thesis, still require a continuum of option prices to calculate the non-central moments. Hence, the implied moment estimators also become biased when option price availability is limited. Specifically, as summarised by Jiang and Tian (2005), the bias is induced by two factors: 1) the discreteness of the strike prices for which option prices are available, and 2) the complete unavailability of option prices for a part of the DOTM-DITM region of the strike price domain, which is also known as truncation.

Given that most of the empirical studies employ interpolation techniques to approximate a continuum of option prices from the discrete set of option prices that are observed from the market, it can be argued that it is relatively easy to address the issue of strike price discreteness. In contrast, it is more difficult to resolve the issue of truncation because the available information is much more limited. As shown in this thesis, the existing truncation error reduction methods, namely LE and DSym, only partially or conditionally reduce the impact of truncation, which can be a serious issue when one is interested in estimating the skewness and kurtosis of the implied RND. This is because these higher moments are more closely related to the tail shape of the density, which is implied by the DOTM option prices.

This thesis demonstrates that truncation can be problematic for implied higher moment estimation when the estimators of Bakshi et al. (2003) are employed, and so develops an alternative truncation treatment method. The new method, DStab, is found to render the truncation error size less volatile either cross-sectionally or over time. It therefore alleviates the de facto effect of truncation on empirical analyses, since most studies utilise the implied moment estimators to track the time series dynamics of the implied moments or else make a cross-sectional comparison of the moments among the options on different underlying assets. With DStab, we suggest that minimising the volatility of the truncation error can be a feasible alternative to minimising the size.

The findings of this thesis could be more meaningful when applied to individual equity options markets. When compared to the S&P 500 index option market that is the main focus of this thesis, options on individual stocks are less liquid and, therefore, their market quotes are only available for a more limited number of strike prices. Hence, it is likely that the issues regarding LE and DSym will be more evident when they are used in conjunction with the individual equity options data, and so it would be interesting to investigate whether DStab renders the truncation error size less volatile even when the method is applied to individual equity options data.

Another topic that can be considered for further research is the effectiveness of DStab when the option price availability is extremely limited. As shown in Chapter 4, it could be very difficult to control the size of the truncation error when the integration domain is so significantly narrow that the estimation bias is large even for the fair value of the volatility contract V. If this is the case, it might be better to discard the observations for the corresponding time and maturity pair rather than applying a truncation treatment method. Hence, it would be valuable if future studies were to determine up to which truncation level DStab can be applied in a reliable manner.

Finally, it would also be meaningful to investigate whether there is another means of implied volatility curve extrapolation that can be shown to be more effective than LE. Although it is natural to think about the approximation of higher orders, for instance linear or quadratic approximation as alternative extrapolation schemes, it is difficult to conclude that they are superior to LE given that the truncation error size is not measurable for the observed market option prices data. However, as shown in Chapter 3, the sensitivity of the implied moment estimators to a marginal change in truncation can be utilised to evaluate the effectiveness of an extrapolation method. Therefore, one could suggest that linear or quadratic approximation should be used instead of LE if it can be shown that these higher order approximations make the implied moment estimators even less sensitive to truncation.

Appendices

A. Proof of Proposition 2.1

Proof. For any nonnegative real constants $K_{\min}$ and $K_{\max}$ which satisfy $0 < K_{\min} \leq \bar{S}$ and $\bar{S} \leq K_{\max} < \infty$, Equation (2.1) in Section 2.2.1 can be rearranged as

$$\begin{aligned} H[S] &= H[\bar{S}] + (S-\bar{S})H_S[\bar{S}] + \mathbb{I}_{S>\bar{S}} \int_{\bar{S}}^{K_{\max}} H_{SS}[K](S-K)dK \\ &\quad + \mathbb{I}_{S>K_{\max}} \int_{K_{\max}}^{\infty} H_{SS}[K](S-K)dK + \mathbb{I}_{S<\bar{S}} \int_{K_{\min}}^{\bar{S}} H_{SS}[K](K-S)dK \\ &\quad + \mathbb{I}_{S<K_{\min}} \int_{0}^{K_{\min}} H_{SS}[K](K-S)dK \\ &= H[\bar{S}] + (S-\bar{S})H_S[\bar{S}] + \int_{\bar{S}}^{K_{\max}} H_{SS}[K](S-K)^+dK \\ &\quad + \int_{K_{\max}}^{\infty} H_{SS}[K](S-K)^+dK + \int_{K_{\min}}^{\bar{S}} H_{SS}[K](K-S)^+dK \\ &\quad + \int_{0}^{K_{\min}} H_{SS}[K](K-S)^+dK, \end{aligned} \tag{1}$$

and, therefore, Equation (2.2) in Section 2.2.1 can also be redefined as

$$\begin{aligned} \mathbb{E}_t^*\{e^{-r\tau}H[S]\} &= (H[\bar{S}] - \bar{S}H_S[\bar{S}])e^{-r\tau} + H_S[\bar{S}]S(t) \\ &\quad + \int_{\bar{S}}^{K_{\max}} H_{SS}[K]C(t,\tau;K)dK + \int_{K_{\max}}^{\infty} H_{SS}[K]C(t,\tau;K)dK \\ &\quad + \int_{K_{\min}}^{\bar{S}} H_{SS}[K]P(t,\tau;K)dK + \int_{0}^{K_{\min}} H_{SS}[K]P(t,\tau;K)dK. \end{aligned} \tag{2}$$

If truncation exists for strike price domains $(0, K_{\min})$ and $(K_{\max}, \infty)$, and therefore, the OTM option prices for the domains are omitted for fair value estimation, then it is equivalent to assuming that $P(t,\tau;K) \equiv 0$ for $\{K : 0 < K < K_{\min}(t,\tau)\}$ and $C(t,\tau;K) \equiv 0$ for $\{K : K_{\max}(t,\tau) < K < \infty\}$, and thereby that

$$\int_{0}^{K_{\min}(t,\tau)} P(t,\tau;K)dK = 0; \quad \text{and} \quad \int_{K_{\max}(t,\tau)}^{\infty} C(t,\tau;K)dK = 0. \tag{3}$$

Hence the assumption of zero option price is equivalent to assuming that

$$\int_{0}^{K_{\min}} H_{SS}[K]P(t,\tau;K)dK = 0; \quad \text{and} \quad \int_{K_{\max}}^{\infty} H_{SS}[K]C(t,\tau;K)dK = 0$$

in Equation (2), and this is also equivalent to assuming that

$$\int_0^{K_{\min}} H_{SS}[K](K-S)^+ dK = 0; \quad \text{and} \quad \int_{K_{\max}}^{\infty} H_{SS}[K](S-K)^+ dK = 0$$

or

$$\mathbb{I}_{S<K_{\min}} \int_0^{K_{\min}} H_{SS}[K](K-S) dK = 0; \quad \text{and} \quad \mathbb{I}_{S>K_{\max}} \int_{K_{\max}}^{\infty} H_{SS}[K](S-K) dK = 0 \quad (4)$$

in Equation (1).

Because $H_{SS}[K]$, $K-S$, and $S-K$ are well-defined for all $H[S]$ and K, the only way to make the assumption in Equation (4) is to assume that both $\mathbb{I}_{S<K_{\min}}$ and $\mathbb{I}_{S>K_{\max}}$ is almost surely zero in the probability measure for which the fair value is estimated, i.e.,

$$\mathbb{P}^*\{S < K_{\min}\} = 0; \quad \text{and} \quad \mathbb{P}^*\{S > K_{\max}\} = 0. \quad (5)$$

If we add the parameters t and τ and rearrange the inequality in terms of log-moneyness and log-return, then Equation (5) becomes identical to Equation (2.19). $\square$

B. Proof of Proposition 3.1

Proposition 3.1 is proved only for the volatility contract V. The same proof can be applied to the other contracts W and X as well. It is assumed that day t, time to maturity τ, and risk-free rate r are known and fixed, and the underlying asset pays no dividends.

Proof.

$$
\begin{aligned}
\widetilde{V}'(\alpha) &= g'(\alpha)\frac{2(1-g(\alpha))}{Se^{g(\alpha)}}C(g(\alpha)) + f'(\alpha)\frac{2(1-f(\alpha))}{Se^{f(\alpha)}}P(f(\alpha)) \\
&\quad - g'(\alpha)\frac{2(1-g(\alpha))}{Se^{g(\alpha)}}\widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), g(\alpha)) - f'(\alpha)\frac{2(1-f(\alpha))}{Se^{f(\alpha)}}\widetilde{P}(\sigma_{\mathrm{BS}}(f(\alpha)), f(\alpha)) \\
&\quad + \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda \\
&\quad + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda \\
&= \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda \\
&\quad + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda)}{\partial \alpha}\right) d\lambda \\
&= \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)}{\partial \sigma_{\mathrm{BS}}(g(\alpha))} \cdot \frac{d\sigma_{\mathrm{BS}}(g(\alpha))}{dg(\alpha)} \cdot \frac{dg(\alpha)}{d\alpha}\right) d\lambda \\
&\quad + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \frac{\partial \widetilde{C}(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda)}{\partial \sigma_{\mathrm{BS}}(f(\alpha))} \cdot \frac{d\sigma_{\mathrm{BS}}(f(\alpha))}{df(\alpha)} \cdot \frac{df(\alpha)}{d\alpha}\right) d\lambda \\
&= \int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda) \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha)\right) d\lambda \\
&\quad + \int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda) \cdot \sigma'_{\mathrm{BS}}(f(\alpha)) \cdot f'(\alpha)\right) d\lambda,
\end{aligned} \tag{6}
$$

where

$$\widetilde{C}(\sigma, \lambda) = S(N(d_1(\sigma, \lambda)) - e^{-\lambda r \tau} N(d_2(\sigma, \lambda))), \tag{7}$$

$$\widetilde{P}(\sigma, \lambda) = S(e^{-\lambda r \tau} N(-d_2(\sigma, \lambda)) - N(-d_1(\sigma, \lambda))), \tag{8}$$

$$d_1(\sigma, \lambda) = \frac{-\lambda + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{9}$$

$$d_2(\sigma, \lambda) = d_1(\sigma, \lambda) - \sigma\sqrt{\tau}, \tag{10}$$

$N(\cdot)$ denotes the standard normal cumulative distribution function, and $\nu(\sigma, \lambda)$ is the option vega for Black-Scholes implied volatility σ and log-moneyness λ. Given that

$$\nu(\sigma, \lambda) = Sn(d_1)\sqrt{\tau}, \tag{11}$$

where

$$n(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \tag{12}$$

$$d_1 = \frac{-\lambda + (r + 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{13}$$

it can be obtained that

$$\nu(\sigma, \lambda) = \frac{S\sqrt{\tau}}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\lambda^2 - 2\lambda(r + 0.5\sigma^2)\tau + (r + 0.5\sigma^2)^2\tau^2}{\sigma^2\tau}\right)\right], \tag{14}$$

and therefore the first integration term in Equation (6) can be rearranged as

$$\begin{aligned}
&\int_{g(\alpha)}^{\lambda_{\max}} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda) \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha)\right) d\lambda \\
&= \frac{1}{S} \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha) \cdot \int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda)e^{-\lambda} \cdot \nu(\sigma_{\mathrm{BS}}(g(\alpha)), \lambda)\right) d\lambda \\
&= \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha) \\
&\quad \cdot \int_{g(\alpha)}^{\lambda_{\max}} (2(1-\lambda) \\
&\quad \exp\left[-\lambda - \frac{1}{2}\left(\frac{\lambda^2 - 2\lambda(r + 0.5\sigma_{\mathrm{BS}}(g(\alpha))^2)\tau + (r + 0.5\sigma_{\mathrm{BS}}(g(\alpha))^2)^2\tau^2}{\sigma_{\mathrm{BS}}(g(\alpha))^2\tau}\right)\right]\Bigg) d\lambda \\
&= \gamma_0 \int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda) \exp\left[\gamma_1\lambda^2 + \gamma_2\lambda + \gamma_3\right]\right) d\lambda,
\end{aligned} \tag{15}$$

where

$$\gamma_0 = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\mathrm{BS}}(g(\alpha)) \cdot g'(\alpha), \tag{16}$$

$$\gamma_1 = -\frac{1}{2\sigma_{\mathrm{BS}}(g(\alpha))^2\tau}, \tag{17}$$

$$\gamma_2 = \frac{r}{\sigma_{\mathrm{BS}}(g(\alpha))^2} - \frac{1}{2}, \tag{18}$$

$$\gamma_3 = -\frac{(r + 0.5\sigma_{\mathrm{BS}}(g(\alpha))^2)^2\tau}{2\sigma_{\mathrm{BS}}(g(\alpha))^2}. \tag{19}$$

Similarly, the second integration term in Equation (6) can be rearranged as

$$\begin{aligned}
&\int_{\lambda_{\min}}^{f(\alpha)} \left(\frac{2(1-\lambda)}{Se^{\lambda}} \cdot \nu(\sigma_{\mathrm{BS}}(f(\alpha)), \lambda) \cdot \sigma'_{\mathrm{BS}}(f(\alpha)) \cdot f'(\alpha) \right) d\lambda \\
&= \delta_0 \int_{\lambda_{\min}}^{f(\alpha)} \left(2(1-\lambda) \exp\left[\delta_1 \lambda^2 + \delta_2 \lambda + \delta_3 \right] \right) d\lambda, \qquad (20)
\end{aligned}$$

where

$$\delta_0 = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot \sigma'_{\mathrm{BS}}(f(\alpha)) \cdot f'(\alpha), \qquad (21)$$

$$\delta_1 = -\frac{1}{2\sigma_{\mathrm{BS}}(f(\alpha))^2 \tau}, \qquad (22)$$

$$\delta_2 = \frac{r}{\sigma_{\mathrm{BS}}(f(\alpha))^2} - \frac{1}{2}, \qquad (23)$$

$$\delta_3 = -\frac{(r + 0.5\sigma_{\mathrm{BS}}(f(\alpha))^2)^2 \tau}{2\sigma_{\mathrm{BS}}(f(\alpha))^2}. \qquad (24)$$

By combining Equations (6), (15), and (20), it can be found that

$$\begin{aligned}
\widetilde{V}'(\alpha) = \gamma_0 &\int_{g(\alpha)}^{\lambda_{\max}} \left(2(1-\lambda) \exp\left[\gamma_1 \lambda^2 + \gamma_2 \lambda + \gamma_3 \right] \right) d\lambda \\
&+ \delta_0 \int_{\lambda_{\min}}^{f(\alpha)} \left(2(1-\lambda) \exp\left[\delta_1 \lambda^2 + \delta_2 \lambda + \delta_3 \right] \right) d\lambda, \qquad (25)
\end{aligned}$$

which is identical to Equation (3.30). □

Bibliography

Aït-Sahalia, Y. and Lo, A. W., 1998, Nonparametric estimation of state-price densities implicit in financial asset prices, *Journal of Finance* 53, 499–547.

Backus, D., Foresi, S. and Wu, L., 2004, Accounting for biases in Black-Scholes, Working paper, New York University.

Bakshi, G., Cao, C. and Chen, Z., 1997, Empirical performance of alternative option pricing models, *Journal of Finance* 52, 2003–2047.

Bakshi, G., Kapadia, N. and Madan, D., 2003, Stock return characteristics, skew laws, and the differential pricing of individual equity options, *Review of Financial Studies* 16, 101–143.

Bali, T. G. and Murray, S., 2013, Does risk-neutral skewness predict the cross section of equity option portfolio returns?, *Journal of Financial and Quantitative Analysis* 48, 1145–1171.

Bliss, R. R. and Panigirtzoglou, N., 2000, Testing the stability of implied probability density functions, Working paper, Bank of England.

Breeden, D. T. and Litzenberger, R. H., 1978, Prices of state-contingent claims implicit in option prices, *Journal of Business* 51, 621–651.

Britten-Jones, M. and Neuberger, A., 2000, Option prices, implied price processes, and stochastic volatility, *Journal of Finance* 55, 839–866.

Broadie, M., Chernov, M. and Johannes, M., 2007, Model specification and risk premia: Evidence from futures options, *Journal of Finance* 62, 1453–1490.

Brown, M. B. and Forsythe, A. B., 1974, Robust tests for the equality of variances, *Journal of the American Statistical Association* 69, 364–367.

Buss, A. and Vilkov, G., 2012, Measuring equity risk with option-implied correlations, *Review of Financial Studies* 25, 3113–3140.

Campa, J. M., Chang, P. H. K. and Reider, R. L., 1998, Implied exchange rate distributions: evidence from otc option markets, *Journal of International Money and Finance* 17, 117–160.

Carr, P. and Madan, D., 2001, Optimal positioning in derivative securities, *Quantitative Finance* 1, 19–37.

Chang, B. Y., Christoffersen, P. and Jacobs, K., 2013, Market skewness risk and the cross section of stock returns, *Journal of Financial Economics* 107, 46–68.

Chang, B. Y., Christoffersen, P., Jacobs, K. and Vainberg, G., 2012, Option-implied measures of equity risk, *Review of Finance* 16, 385–428.

Chou, R. K., Chung, S. L., Hsiao, Y. J. and Wang, Y. H., 2011, The impact of liquidity on option prices, *Journal of Futures Markets* 31, 1116–1141.

Christensen, B. J., Hansen, C. S. and Prabhala, N. R., 2002, The telescoping overlap problem in options data, Working paper, University of Aarhus and University of Maryland.

Conrad, J., Dittmar, F. and Ghysels, E., 2013, Ex ante skewness and expected stock returns, *Journal of Finance* 68, 85–124.

Corrado, C. J. and Su, T., 1997, Implied volatility skews and stock return skewness and kurtosis implied by stock option prices, *European Journal of Finance* 3, 73–85.

DeMiguel, V., Plyakha, Y., Uppal, R. and Vilkov, G., 2013, Improving portfolio selection using option-implied volatility and skewness, *Journal of Financial and Quantitative Analysis* 48, 1813–1845.

Dennis, P. and Mayhew, S., 2002, Risk-neutral skewness: Evidence from stock options, *Journal of Financial and Quantitative Analysis* 37, 471–493.

Dennis, P. and Mayhew, S., 2009, Microstructural biases in empirical tests of option pricing models, *Review of Derivatives Research* 12, 169–191.

Diavatopoulos, D., Doran, J. S., Fodor, A. and Peterson, D. R., 2012, The information content of implied skewness and kurtosis changes prior to earnings announcements for stock and option returns, *Journal of Banking and Finance* 36, 786–802.

Doran, J. S. and Krieger, K., 2010, Implications for asset returns in the implied volatility skew, *Financial Analysts Journal* 66, 65–76.

Duan, J. C. and Wei, J., 2009, Systematic risk and the price structure of individual equity options, *Review of Financial Studies* 22, 1981–2006.

Gemmill, G. and Saflekos, A., 2000, How useful are implied distributions? evidence from stock index options, *Journal of Derivatives* 7, 83–91.

Han, B., 2008, Investor sentiment and option prices, *Review of Financial Studies* 21, 387–414.

Jackwerth, J. C. and Rubinstein, M., 1996, Recovering probability distributions from option prices, *Journal of Finance* 51, 1611–1631.

Jiang, G. J. and Tian, Y. S., 2005, The model-free implied volatility and its information content, *Review of Financial Studies* 18, 1305–1342.

Kozhan, R., Neuberger, A. and Schneider, P., 2013, The skew risk premium in the equity index market, *Review of Financial Studies* 26, 2174–2203.

Levene, H., 1960, Robust tests for equality of variances, in I. Olkin, S. G. Ghurye, W. Hoeffding, W. G. Madow, and H. B. Mann, ed.: *Probability and Statistics: Essays in Honor of Harold Hotelling* (Stanford University Press: Menlo Park, CA).

Lin, B. H., Chang, I. J. and Paxson, D. A., 2008, Smiling less at LIFFE, *Journal of Futures Markets* 28, 57–81.

Malz, A. A., 1997, Estimating the probability distribution of the future exchange rate from option prices, *Journal of Derivatives* 5, 18–36.

Neumann, M. and Skiadopoulos, G., 2013, Predictable dynamics in higher order risk-neutral moments: Evidence from the S&P 500 options, *Journal of Financial and Quantitative Analysis* 48, 947–977.

Shimko, D., 1993, Bounds on probability, *Risk* 6, 33–37.

Tian, Y., 2011, Extracting risk-neutral density and its moments from american option prices, *Journal of Derivatives* 18, 17–34.

Xing, Y., Zhang, X. and Zhao, R., 2010, What does the individual option volatility smirk tell us about future equity returns?, *Journal of Financial and Quantitative Analysis* 45, 641–662.

Zhang, J. E. and Xiang, Y., 2008, The implied volatility smirk, *Quantitative Finance* 8, 263–284.